TRANSPORT NOISE ARISING FROM DIFFUSION AND BULK OR SURFACE GENERATION-RECOMBINATION IN SEMICONDUCTORS

BY

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To

Dear mother and (late) father

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TABLE OF CONTENTS

ACKNOWLEDGEMENTS		iii
ABSTRACT		vii
Chapter		Page
1	INTRODUCTION	1
	1.1 Object	1
	1.2 Flicker Noise	1
	1.3 Transport Noise	2
	1.4 Surface Noise	4
	1.5 Embedded and Nonembedded Bodies	5
PART A: MATHEMATICAL METHODS AND PHYSICAL NOISE SOURCES		8
2	MATHEMATICAL METHODS FOR TRANSPORT NOISE	8
	2.1 Methods	8
	2.2 Langevin Equation and Source Methods	9
	2.3 The Phenomenological Equation and Correlation Methods	19
3	PHYSICAL NOISE SOURCES	28
	3.1 Introduction	28
	3.2 Volume Noise Sources	28
	3.3 Surface Noise Sources	34

Chapter		Page
PART B:	NOISE DUE TO VOLUME NOISE SOURCES	37
4	SYMMETRICAL EMBEDDED BODIES	37
	4.1 Introduction	37
	4.2 Survey of Older Work	37
	4.3 Previous Work Using the Green's Function Method	39
	4.4 Present Work: Carrier Fluctuations in Sphere Due to Diffusion and Generation-Recombination	42
5	NONSYMMETRICAL EMBEDDED BODIES	52
	5.1 Introduction	52
	5.2 Voss and Clarke's 'P' Sources	52
	5.3 Physical Diffusion Source	58
6	SYMMETRICAL NONEMBEDDED BODIES	68
	6.1 Introduction	68
	6.2 Summary of Previous Work	68
	6.3 Present Work: Sphere	71
7	NONSYMMETRICAL NONEMBEDDED BODIES	90
	7.1 Introduction	90
	7.2 Linear Case	90
	7.3 Rectangular Nonembedded Bar	92
PART C:	NOISE DUE TO SURFACE NOISE SOURCES	98
8	NOISE DUE TO STOCHASTIC BOUNDARY CONDITIONS	98
	8.1 Introduction	98
	8.2 Summary of Previous Work	98
	8.3 Noise Caused by Surface Generation-Recombination	99

APPENDIX 119

REFERENCES 128

BIOGRAPHICAL SKETCH 130

Abstract of Dissertation Presented to the Graduate Council
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Major department: Electrical Engineering

The purpose of this thesis is to present a systematic study of the type of noise spectra in solids or solid state devices, which can arise from "transport noise," which stands for noise due to combination of causes, such as diffusion of carriers, generation and recombination (g-r) processes of carriers in the bulk and at the surface. The techniques to be described apply equally well to the problem of heat diffusion, with heat transfer or reflection at the boundaries of the system; we have, however, mainly the carrier noise problem in mind.

After an introduction we describe the various mathematical techniques, applicable to transport noise. The methods studied are (1) the eigenfunctions expansion method, (2) the Green's function method. Each of these methods can be applied to two physically distinct procedures, (a) the sources or Schottky method, (b) the correlation method.

In chapter 3 we discuss the physical noise sources,viz. the diffusion source, g-r source and surface source, as well as Clarke and Voss' P-source.

In the next few chapters we discuss: transport noise in symmetrical embedded bodies, in nonsymmetrical embedded bodies, in symmetrical non-embedded bodies and in nonsymmetrical nonembedded bodies. The concept "embedded" body refers to a system which is part of a very large system, with which it exchanges particles or heat, such as a volume between probes on a large semiconductor. The concept "nonembedded body" refers to a body inside which there is transport noise, with deterministic boundary conditions on the surface of the body.

In chapter 4 we first review the older work on the circular patch and the cylinder (MacFarlane, Burgess, Richardson, van Vliet, Chenette). Next we solve for the noise of an embedded sphere, caused by diffusion and volume g-r processes. For infinite carrier lifetime, the results are found to be in agreement with those of Fassett. For finite lifetime an extension of his results is obtained. The high frequency $\bar{\omega}^{3/2}$ law is confirmed.

In chapter 5 we discuss the results for a rectangular embedded bar, disc, or cube. Because of the lesser symmetry, the spectrum cannot be obtained in closed form, but only in a triple integral form. The result has been numerically evaluated by computer techniques. Not all the ranges predicted by Clarke and Voss are visible, however. For the P-source the noise has also been evaluated. Contrary to Clarke

and Voss' claim, no 1/f range is visible. Thus, their heat diffusion noise model cannot account for the observed 1/f noise in metal films.

In chapter 6 symmetrical nonembedded bodies are discussed. Particular attention is given to the noise in a sphere, with boundary conditions due to surface recombination. A closed form is obtained via an expansion of the Green's function in spherical polynomials.

In chapter 7 the nonsymmetrical bar is discussed. The results given previously by Lax and Mengert and others were evaluated by computer.

In chapter 8, finally, we discuss noise due to the surface g-r source. This noise can be dominant due to the surface modulation factor; it has not been considered in all older theories. We discuss three examples: (a) a semiconductor single crystal sample with surface g-r noise; (b) the noise in the channel of a MOSFET, caused by diffusion into the adjacent bulk and by g-r (surface) processes near the oxide layer; (c) the same model but with special care for the alteration of the diffusion constant and mobility in the channel due to surface scattering. For models (b) and (c) it is shown that a realistic large 1/f range (10^4 hz up to 10^{11} hz) can occur. Thus this model can account for much of the high frequency 1/f noise in MOS devices.

The other examples and models considered in this thesis firmly indicate that volume g-r and diffusion processes give rather smooth spectra, but without a 1/f range. The high frequency asymptote is always $\bar{\omega}^{3/2}$ (embedded case) or $\bar{\omega}^{2}$ (nonembedded case).

CHAPTER 1
INTRODUCTION

1.1 Object

The purpose of this study is to make a systematical investigation of diffusion and generation-recombination noise spectra. Such studies may aid in understanding the physical mechanisms underlying '1/f' noise. The noise spectra obtainable from transport processes are analyzed and the possibility of contributions to '1/f' noise is studied. It will be shown that most transport processes caused by volume sources do not give rise to 1/f noise.

In addition, a detailed investigation has been undertaken of surface noise. This is noise caused by surface sources which give rise to stochastic boundary conditions at the surface. Surface noise contributions are studied in detail, since the older theories due to Hyde[1], Champlin[2], Lax-Mengert[3],and van Vliet[4] only considered the volume noise contribution. This analysis gives for the first time a transport noise model which results in 1/f noise; however, the frequency range for which it occurs is often too high to account for the experimental data, unless very long lifetimes of the carriers can be shown to exist.

1.2 Flicker Noise

The phenomenon of 1/f noise or 'flicker noise' started with the study of J.B.Johnson[5] on noise in vacuum tubes in 1925. He

noted, that besides shot noise, which has a white frequency spectrum, at low frequencies another noise component appears, which varies roughly inversely proportional to the frequency. The first interpretation of this noise was given by W.Schottky[6] in 1926. Schottky attributed the effect to fluctuations in the work function of the cathode material, caused by the random arrival and leaving of foreign atoms at the cathode surface. The spectrum computed by Schottky was not exactly of the form $S(f) = A/f$; rather, his effect results in a Lorentzian spectrum, $S(f) = B/(1+\omega^2\tau^2)$.

Since Schottky's mechanism failed to interpret Johnson's 1/f data, as well as the flicker noise later on observed in a large number of solids and devices, various authors have proposed a great variety of detailed studies to understand the physical mechanisms behind 1/f noise. In the subsequent chapters we shall summarize some of these efforts and extensively study the contribution of transport and surface noise to the origin of flicker noise.

1.3 Transport Noise

Transport noise is caused by quantities subject to stochastic transport in the sample. The most notable examples involve particle diffusion and heat diffusion. Both are described by a linear partial differential equation. The noise can be obtained in two ways. In the first procedure, the 'sources' or 'Langevin' procedure, one adds

noise source terms to the phenomenological transport equation, which then becomes of the form

$$\left(\frac{\partial}{\partial t} + \Lambda\right) a(\underset{\sim}{r},t) = \xi(\underset{\sim}{r},t) \tag{1.3.1}$$

where $\Lambda = -D\nabla^2$ and $a = \Delta T$ or Δn for heat or particle diffusion respectively, and ξ is the volume noise source. The spectrum of the latter must be known.

In the correlation procedure, on the other hand, one uses the ensemble-averaged stochastic equation, which by Onsager's[7] principle, is the phenomenological equation. Thus we must then solve

$$\left(\frac{\partial}{\partial t} + \Lambda\right) \langle a(\underset{\sim}{r},t) | a(\underset{\sim}{r},0)\rangle = 0 \tag{1.3.2}$$

where $\langle | \rangle$ denotes the ensemble average of $a(\underset{\sim}{r},t)$, conditional to a given value $a(\underset{\sim}{r},0)$ at time $t = 0$. In this case the covariance function

$$\Gamma(\underset{\sim}{r},\underset{\sim}{r}') = \langle \Delta a(\underset{\sim}{r},t)\Delta a(\underset{\sim}{r}',t)\rangle \tag{1.3.3}$$

must be known.

The connection between both procedures is given by Λ-theorem due to van Vliet[8], which connects the source spectra to the covariance function

$$(\Lambda_{\underset{\sim}{r}'} + \Lambda_{\underset{\sim}{r}})\Gamma(\underset{\sim}{r},\underset{\sim}{r}') = \frac{1}{2}S_{\xi}(\underset{\sim}{r},\underset{\sim}{r}'). \tag{1.3.4}$$

To solve either (1.3.1) or (1.3.2) one can use two mathematical formalisms, viz. the eigenfunction expansion method and the Green's

function description. Thus, in total, there are four methods to obtain transport noise. The equivalence of all these methods has been shown by van Vliet and Fassett[9,10]. The details of these methods are described in the next chapter.

1.4 Surface Noise

This is noise caused by stochastic boundary conditions at the surface. An example occurs when a semiconductor sample is subject to surface generation-recombination, with the surface recombination velocity S being unequal to zero or infinity.

In the experiments on modulation of conductance by surface charges, it was found that only 10 percent of the induced charges were effective in changing the conductance. J.Bardeen[11] proposed that the ineffective portion of the induced charges is lodged in states localized at the surface. Since the work of McWhorter[12] and others, it is held that the traps in the oxide modulate the generation-recombination rate and this gives rise to flicker noise in the device. Hooge[13] explained the excess noise by an empirical model, though in his case the noise is a volume rather than a surface effect.

Recently van Vliet, van der Ziel and Schmidt[14] considered heat conduction in a thin film supported by a substrate and found that surface sources can give rise to 1/f noise. A sample with a small inversion layer near the surface which is coupled to larger

bulk volume is considered in this thesis. The model is mathematically similar to that considered by van Vliet et al.,[14] and, as for the thin film case, the surface noise gives rise to 1/f type of noise behavior with some theoretical restrictions.

1.5 Embedded and Non-embedded Bodies

From the theory it emerges that one must distinguish between models involving embedded and non-embedded domains. In the former case, one supposes that the volume of interest for the noise, V_s, is part of a larger domain V. Fluctuations in the particle number or heat content of V_s arise due to sinks and sources in V_s (g-r contributions) and due to transport across the boundary into the rest of V (diffusion-drift contributions). An example is afforded by the case of field emission from a metal tip. Here ions within a, say circular, patch of the active area tip can modulate the electron emission; these ions therefore affect the emission, when they are within this circular patch V_s, which is a part of the larger cathode area V over which ions diffuse.

The non-embedded case applies when we monitor the particle or heat content of a domain V_s which does not communicate with the exterior but on the boundary of which we have 'deterministic boundary conditions', usually of the Dirichlet, Neumann or mixed type. An example is when we measure noise due to carrier fluctuations in a semiconductor sample V_s, subject to surface generation-recombination (mixed boundary conditions) on the free surface.

This dissertation is divided into three parts. The first part, A, contains the mathematical and physical basis; in particular in chapters 2 and 3 we discuss the various mathematical methods for transport noise and the physical noise sources, respectively. The second part, B, gives the noise due to volume sources. In chapter 4 we discuss the noise for symmetrical embedded bodies. Chapter 5 is devoted to noise from nonsymmetrical embedded bodies. Chapters 6 and 7 consider the noise from symmetrical and nonsymmetrical non-embedded bodies, respectively. Finally, in the third and last part, C, we consider the noise due to stochastic boundary conditions. In chapter 8 we study in detail the noise caused by surface generation-recombination processes in bulk samples as well as in MOSFETs.

PART A
MATHEMATICAL METHODS AND PHYSICAL NOISE SOURCES

CHAPTER 2
MATHEMATICAL METHODS FOR TRANSPORT NOISE

2.1 Methods

As indicated in the introduction, there are basically two ways to attack the problem of transport noise. The first class of procedures considers (a) transport equation(s) in the sense of Langevin, that is, one adds source terms $\xi(\underset{\sim}{r},t)$ which themselves have no memory, to the phenomenological equation(s) for the fluctuating variable(s) $a(\underset{\sim}{r},t)$; the variables $a(\underset{\sim}{r},t)$ are Markovian random of infinite dimensionality since $\underset{\sim}{r}$ is a continuous parameter like a position in some domain V. The boundary conditions on the surface S of V may be deterministic or stochastic; in the latter case surface Langevin sources $\underset{\sim}{\zeta}(\underset{\sim}{r}_o,t)$ must be specified ($\underset{\sim}{r}_o \in S$). The "Langevin procedure" or "sources procedure" is still the most versatile one. The stochastic Langevin equation can be solved with Green's functions, orthogonal or biorthogonal expansions or Fourier methods.

The second class of procedures is based on the solution of the transport equation for the ensemble averaged transport quantities $\langle a(r,t)|a(r,0)\rangle \equiv \langle a(\underset{\sim}{r},t)\rangle_{a(\underset{\sim}{r}_o)}$, in which the fluctuations $\Delta a(\underset{\sim}{r},t=0)$ are specified. The solution involves the "correlation procedure." The noise is obtained by the Fourier-Laplace transformed Green's function of the transport equation, providing the covariance function, $\Gamma(\underset{\sim}{r},\underset{\sim}{r}') = \langle \Delta a(\underset{\sim}{r},t)\Delta a(\underset{\sim}{r}',t)\rangle_s$ is known;

here $\langle\ \rangle_s$ denotes a stationary nonconditional average. This leads to the form suggested by van Vliet and Fassett[9]. This procedure is mathematically much faster than the Langevin procedure; however, it has the disadvantage that stochastic boundary conditions cannot be included. Nevertheless, if the transport equation is linear and if the surface sources are uncorrelated with the volume sources, the former can be separately considered with the sources method, due to the superposition principle. Also, despite the fact that the covariance function in the presence of surface sources is often not unique, it has been shown by van Vliet[10] that the spectra resulting from the Green's function procedure are unique. We will discuss both classes of methods and their prospects for obtaining 1/f noise.

2.2 Langevin Equation and Source Methods

The stochastic transport equation is supposed to be of the form

$$La(\underset{\sim}{r},t) = \left(\frac{\partial}{\partial t} + \Lambda_{\underset{\sim}{r}}\right)a(\underset{\sim}{r},t) = \xi(\underset{\sim}{r},t), \tag{2.2.1}$$

where L is a linear differential or integral operator. For the case of diffusion:

$$\Lambda = -D\nabla^2, \begin{cases} a = \Delta n & \text{(particle diffusion)} \\ a = \Delta T & \text{(heat diffusion)} \end{cases} \tag{2.2.2}$$

In the diffusion model Λ is a self adjoint operator which simplifies the mathematics considerably. The boundary conditions must be specified at the boundary S of V. The noise may concern the entire

domain V or subdomain $V_s \subset V$. If V is infinite, the only condition is that $a(\underset{\sim}{r},t)$ goes sufficiently fast to zero. If on the surface S of V we have

$$ma(\underset{\sim}{r}_o,t) = 0, \quad \underset{\sim}{r}_o \in S, \tag{2.2.3}$$

m being a linear surface operator, the boundary conditions are deterministic; if

$$ma(\underset{\sim}{r}_o,t) = \zeta(\underset{\sim}{r}_o,t), \quad \underset{\sim}{r}_o \in S, \tag{2.2.4}$$

the boundary conditions are stochastic.

2.2.1 The Bulk Noise

Because of the superposition principle, noise associated with ξ and ζ can be separately considered. Thus first setting $\zeta = 0$, the boundary conditions are deterministic (2.2.3). Various methods will be given to solve from (2.2.1)-(2.2.3).

2.2.1(a) Eigenfunction expansion

Let $\phi_k(\underset{\sim}{r})$ be the eigenfunctions of Λ subject to b.c.(2.2.3) that is

$$\Lambda\phi_k(\underset{\sim}{r}) = \lambda_k\phi_k(\underset{\sim}{r}); \tag{2.2.5}$$

let $\psi_k(\underset{\sim}{r})$ be the eigenfunctions of the adjoint operator $\Lambda^\dagger$, then

$$\Lambda^\dagger\psi_k(\underset{\sim}{r}) = \lambda_k^*\psi_k(\underset{\sim}{r}). \tag{2.2.6}$$

The functions ϕ and ψ are biorthogonal, i.e.

$$(\phi_k,\psi_\ell) = \int_V d^3r\ \phi_k(\underset{\sim}{r})\psi_\ell^*(\underset{\sim}{r}) = \delta_{k\ell}. \tag{2.2.7}$$

Expanding $a(\underset{\sim}{r},t)$ and $\xi(\underset{\sim}{r},t)$ in the functions ϕ_k and denoting the expansion coefficients by $a_k(t)$ and $\xi_k(t)$, then (2.2.1) gives

$$\sum_k [\dot{a}_k(t)+\lambda_k a_k(t)]\phi_k(\underset{\sim}{r}) = \sum_k \xi_k(t)\phi_k(\underset{\sim}{r}). \tag{2.2.8}$$

Multiplying with $\psi_\ell^*(\underset{\sim}{r})$ and integrating over V and with (2.2.7) one obtains

$$(\frac{d}{dt} + \lambda_\ell)a_\ell(t) = \xi_\ell(t), \tag{2.2.9}$$

which is a standard one variable Langevin equation. In the usual method of solution truncated Fourier series in the time interval (0,T) are used. Fourier integrals are not suitable since the Langevin equation (2.2.9) does not apply for negative T; thus

$$a_\ell(t) = \sum_{n=-\infty}^{\infty} a_{\ell n} e^{i\omega_n t}, \quad \omega_n = 2\pi n/T, \; n=0,\pm 1,\ldots, \tag{2.2.10}$$

and similarly for $\xi_\ell(t)$, one obtains

$$\langle a_{kn} a_{\ell n}^* \rangle_s = \frac{\langle \xi_{kn} \xi_{\ell n}^* \rangle_s}{(i\omega_n+\lambda_k)(-i\omega_n+\lambda_\ell^*)} . \tag{2.2.11}$$

Now the essence of the sources method is that the white spectra of $\langle \xi(\underset{\sim}{r},t)\xi(\underset{\sim}{r}',t)\rangle$, denoted as $S_\xi(\underset{\sim}{r},\underset{\sim}{r}')$, are known. Thus according to the Wiener-Khintchine theorem, the sources correlation function is

$$\langle \xi(\underset{\sim}{r},t)\xi^*(\underset{\sim}{r}',t')\rangle_s = \tfrac{1}{2} S_\xi(\underset{\sim}{r},\underset{\sim}{r}')\delta(t-t') \tag{2.2.12}$$

($\xi=\xi^*$ is real), in which S_ξ must be known. Expanding in E.F., we have

$$\sum_{k\ell} \langle \xi_k(t)\xi_\ell^*(t')\rangle_s \phi_k(\underset{\sim}{r})\phi_\ell^*(\underset{\sim}{r}') = \frac{1}{2}S_\xi(\underset{\sim}{r},\underset{\sim}{r}')\delta(t-t'). \qquad (2.2.13)$$

Multiplying this with $\psi_{k'}^*(\underset{\sim}{r})\psi_{\ell'}(\underset{\sim}{r}')$, integrating over $d^3r d^3r'$, and using biorthogonality, it is found that

$$\langle \xi_k(t)\xi_\ell^*(t')\rangle_s = \frac{1}{2}\iint d^3r d^3r' S_\xi(\underset{\sim}{r},\underset{\sim}{r}')\psi_k^*(\underset{\sim}{r})\psi_\ell(\underset{\sim}{r}')\delta(t-t'). \qquad (2.2.14)$$

Representing the ξ's by truncated Fourier series we further have,

$$\sum_{nm} \langle \xi_{kn}\xi_{\ell m}^*\rangle_s e^{i(\omega_n t-\omega_m t')} = \frac{1}{2T}\iint d^3r d^3r' S_\xi(\underset{\sim}{r},\underset{\sim}{r}')\psi_k^*(\underset{\sim}{r})\psi_\ell(\underset{\sim}{r}')\Sigma e^{i\omega_n(t-t')} \qquad (2.2.15)$$

from which we obtain

$$\langle \xi_{kn}\xi_{\ell m}^*\rangle_s = \frac{\delta_{nm}}{2T}\iint d^3r d^3r' S_\xi(\underset{\sim}{r},\underset{\sim}{r}')\psi_k^*(\underset{\sim}{r})\psi_\ell(\underset{\sim}{r}'). \qquad (2.2.16)$$

The spectrum is given by

$$S_{a_k a_\ell^*}(\omega_n) = \lim_{T\to\infty} 2T\langle a_{kn}a_{\ell n}^*\rangle_s. \qquad (2.2.17)$$

Thus from (2.2.11) and (2.2.16)

$$S_{a_k a_\ell^*}(\omega) = \frac{1}{(i\omega+\lambda_k)(-i\omega+\lambda_\ell^*)}\iint d^3r d^3r' S_\xi(\underset{\sim}{r},\underset{\sim}{r}')\psi_k^*(\underset{\sim}{r})\psi_\ell(\underset{\sim}{r}'). \qquad (2.2.18)$$

Finally for the spectra of $\langle a(\underset{\sim}{r},t)a(\underset{\sim}{r}',t)\rangle$ one obtains for the noise due to the bulk source ξ:

$$S_a^{bulk}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = \sum_{k\ell} S_{a_k a_\ell^*}(\omega)\phi_k(\underset{\sim}{r})\phi_\ell^*(\underset{\sim}{r}'), \qquad (2.2.19)$$

or substituting from (2.2.18) one obtains

$$S_a^{bulk}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = \sum_{k\ell} \frac{\phi_k(\underset{\sim}{r})\phi_\ell^*(\underset{\sim}{r}')}{(\lambda_k+i\omega)(\lambda_\ell^*-i\omega)}\iint d^3r_1 d^3r_2 S_\xi(\underset{\sim}{r}_1,\underset{\sim}{r}_2)\psi_k^*(\underset{\sim}{r}_1)\psi_\ell(\underset{\sim}{r}_2). \qquad (2.2.20)$$

To show the connection with the correlation procedure to be discussed in section 2.3, we write

$$\frac{1}{(\lambda_k+i\omega)(\lambda_\ell^*-i\omega)} = \frac{1}{\lambda_k+\lambda_\ell^*}\left(\frac{1}{\lambda_k+i\omega}+\frac{1}{\lambda_\ell^*-i\omega}\right). \tag{2.2.21}$$

For the covariance from (2.2.20) and (2.2.21)

$$\Gamma^{bulk}(\mathfrak{r},\mathfrak{r}') = \int_0^\infty d\left(\frac{\omega}{2\pi}\right)\mathrm{Re}S_a^{bulk}(\mathfrak{r},\mathfrak{r}',\omega)$$

$$= \frac{1}{2}\sum_{k\ell}\frac{\phi_k(\mathfrak{r})\phi_\ell^*(\mathfrak{r}')}{\lambda_k+\lambda_\ell^*}\iint d^3r_1 d^3r_2 S_\xi(\mathfrak{r}_1,\mathfrak{r}_2)\psi_k^*(\mathfrak{r}_1)\psi_\ell(\mathfrak{r}_2). \tag{2.2.22}$$

Operating on the LHS with $\Lambda_{\mathfrak{r}}+\Lambda_{\mathfrak{r}'}$ we note with (2.2.5)

$$(\Lambda_{\mathfrak{r}}+\Lambda_{\mathfrak{r}'})\Gamma^{bulk}(\mathfrak{r},\mathfrak{r}') = \frac{1}{2}\sum_{k\ell}\iint d^3r_1 d^3r_2 S_\xi(\mathfrak{r}_1,\mathfrak{r}_2)\phi_k(\mathfrak{r})\phi_\ell^*(\mathfrak{r}')\psi_k^*(\mathfrak{r}_1)\psi_\ell(\mathfrak{r}_2). \tag{2.2.23}$$

Observing the closure property

$$\sum_k \phi_k(\mathfrak{r})\psi_k^*(\mathfrak{r}_1) = \delta(\mathfrak{r}-\mathfrak{r}_1) \tag{2.2.24a}$$

$$\sum_\ell \phi_\ell^*(\mathfrak{r}')\psi_\ell(\mathfrak{r}_2) = \delta(\mathfrak{r}'-\mathfrak{r}_2) \tag{2.2.24b}$$

and integrating out the delta functions we find

$$(\Lambda_{\mathfrak{r}}+\Lambda_{\mathfrak{r}'})\Gamma^{bulk}(\mathfrak{r},\mathfrak{r}') = \frac{1}{2}S_\xi(\mathfrak{r},\mathfrak{r}'). \tag{2.2.25}$$

This is the Λ-theorem due to van Vliet[8]. This theorem gives the connection between the sources spectra and the covariance function. Also it warns that sources spectra cannot be postulated ad hoc, if $\Gamma^{bulk}(\mathfrak{r},\mathfrak{r}')$ is known from statistical mechanics.

Substituting (2.2.25) in equation (2.2.20) and using equation (2.2.21) one obtains

$$S_a^{bulk}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = 2\sum_{k\ell}\frac{\phi_k(\underset{\sim}{r})\phi_\ell^*(\underset{\sim}{r}')}{\lambda_k+\lambda_\ell^*}\left(\frac{1}{\lambda_k+i\omega}+\frac{1}{\lambda_\ell^*-i\omega}\right)$$

$$\times \iint d^3r_1 d^3r_2 (\Lambda_{\underset{\sim}{r}_1}+\Lambda_{\underset{\sim}{r}_2})\Gamma^{bulk}(\underset{\sim}{r}_1,\underset{\sim}{r}_2)\psi_k^*(\underset{\sim}{r}_1)\psi_\ell(\underset{\sim}{r}_2). \quad (2.2.26)$$

Introducing Green's theorem for the operators Λ and $\Lambda^\dagger$:

$$(\Lambda f,g)-(f,\Lambda^\dagger g) = \oint C[f,g^*]\cdot d\sigma, \quad (2.2.27)$$

where $\oint C\cdot d\sigma$ is the bilinear concomitant involving a surface integral. Thus in (2.2.26)

$$\int \Lambda_{\underset{\sim}{r}}\Gamma^{bulk}(\underset{\sim}{r}_1,\underset{\sim}{r}_2)\psi_k^*(\underset{\sim}{r}_1)d^3r_1 = \int\Gamma^{bulk}(\underset{\sim}{r}_1,\underset{\sim}{r}_2)\Lambda^\dagger_{\underset{\sim}{r}_1}\psi_k^*(\underset{\sim}{r}_1)d^3r_1$$

$$+\oint d^2r_0 C[\Gamma^{bulk}(\underset{\sim}{r}_0,\underset{\sim}{r}_2),\psi_k^*(\underset{\sim}{r}_0)]$$

$$= \lambda_k^*\int d^3r_1\Gamma^{bulk}(\underset{\sim}{r}_1,\underset{\sim}{r}_2)\psi_k^*(\underset{\sim}{r}_1)$$

$$+\oint d^2r_0 C[\Gamma^{bulk}(\underset{\sim}{r}_0,\underset{\sim}{r}_2),\psi_k^*(\underset{\sim}{r}_0)]. \quad (2.2.28)$$

Doing the same for $\Lambda\underset{\sim}{r}_2$, and keeping in mind the closure property, we get

$$S_a^{bulk}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = 2\sum_k\phi_k(\underset{\sim}{r})\frac{1}{\lambda_k+i\omega}\int d^3r''\Gamma(\underset{\sim}{r}'',\underset{\sim}{r}')\psi_k^*(\underset{\sim}{r}'')$$

$$+ \text{hcj} + \text{surface terms}, \quad (2.2.29)$$

where hcj stands for Hermitian conjugate, $r \leftrightarrow r'$, $i\omega \leftrightarrow -i\omega$, $\lambda \to \lambda^*$; the surface term involves the bilinear concomitant. The first part of (2.2.29) is called the volume contribution. It will be denoted by S_a^{vol}. When the bilinear concomitant is zero - as is often the case - the volume contribution equals the bulk contribution (2.2.29).

Often the interesting part is to know the spectra of the total value or spatial average of variable $a(\underset{\sim}{r},t)$ in V_s. Thus let

$$A(t) = \int_{V_s} a(\underset{\sim}{r},t)d^3r. \tag{2.2.30}$$

The integration yields for (2.2.20)

Sources form:

$$S_A^{bulk}(\omega) = \sum_{k\ell} \int_{V_s} \int_{V_s} d^3r d^3r' \frac{\phi_k(\underset{\sim}{r})\phi_\ell^*(\underset{\sim}{r}')}{(\lambda_k+i\omega)(\lambda_\ell^*-i\omega)}$$

$$\times \int_V \int_V d^3r_1 d^3r_2 S_\xi(\underset{\sim}{r}_1,\underset{\sim}{r}_2)\psi_k^*(\underset{\sim}{r}_1)\psi_\ell(\underset{\sim}{r}_2). \tag{2.2.31}$$

The correlation form, obtainable from (2.2.29) contains $\int d^3r'\Gamma(\underset{\sim}{r}'',\underset{\sim}{r}')$. Let us suppose, as is often the case,

$$\Gamma(\underset{\sim}{r}'',\underset{\sim}{r}') = [\langle \Delta A^2 \rangle / V_s]\delta(\underset{\sim}{r}''-\underset{\sim}{r}'). \tag{2.2.32}$$

Then we find

Correlation form:

$$S_A^{vol}(\omega) = \frac{4\langle \Delta A^2 \rangle}{V_s} \mathrm{Re} \int_{V_s} \int_{V_s} d^3r d^3r' \sum_k \frac{\phi_k(\underset{\sim}{r})\psi_k^*(\underset{\sim}{r}')}{\lambda_k + i\omega} \tag{2.2.33}$$

In general $\lambda_k = \mu_k + i\nu_k$ is complex; for self adjoint case $\nu_k = 0$ and (2.2.33) reduces to

$$S_a^{vol}(\omega) = \frac{4\langle \Delta A^2 \rangle}{V_s} \sum_k \frac{\mu_k \left| \int d^3r \phi_k(\underset{\sim}{r}) \right|^2}{\mu_k^2 + \omega^2} . \tag{2.2.34}$$

Finally, note that the eigenfunction method has several drawbacks. If the domain V is finite, the results occur in a series form which is not easily evaluated. If the eigenfunctions can be taken to be plane waves, then we end up with the integrals over $\underset{\sim}{k}$-space which are not trivial. We therefore prefer the Green's function method, whenever the Green's function can be evaluated in closed form, as is the case for suitable geometries (one dimensional bar, cylinder,sphere). The limiting spectral ranges can then be obtained.

2.2.1(b) Green's function procedure

The Green's function procedure is much more direct. The Langevin's equation (2.2.1) is Fourier transformed with

$$a(\underset{\sim}{r},t) = \sum_n a_n(\underset{\sim}{r})\, e^{i\omega_n t}, \omega_n = \frac{2\pi n}{T}, \; n=0,\pm1,\pm2,\ldots \tag{2.2.35a}$$

$$\xi(\underset{\sim}{r},t) = \sum_n \xi_n(\underset{\sim}{r})\, e^{i\omega_n t} . \tag{2.2.35b}$$

This gives

$$(i\omega_n+\Lambda_{\underset{\sim}{r}})a_n(\underset{\sim}{r}) = \xi_n(\underset{\sim}{r}). \tag{2.2.36}$$

The Green's function $G(\underset{\sim}{r},\underset{\sim}{r}',i\omega_n)$ is defined by

$$(i\omega_n+\Lambda_{\underset{\sim}{r}})G(\underset{\sim}{r},\underset{\sim}{r}',i\omega_n) = \delta(\underset{\sim}{r}-\underset{\sim}{r}'), \tag{2.2.37}$$

and satisfies homogeneous boundary conditions corresponding to equation (2.2.3). Using (2.2.37), the solution of (2.2.36) is found by using the standard procedure

$$a_n(\underset{\sim}{r}) = \int_V d^3r_1 G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega_n)\xi_n(\underset{\sim}{r}_1). \tag{2.2.38}$$

The spectrum is found by the result analogous to (2.2.17)

$$S_a^{bulk}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = \lim_{T\to\infty} 2T\langle a_n(\underset{\sim}{r})a_n^*(\underset{\sim}{r}')\rangle$$

$$= \int_V\int_V d^3r_1 d^3r_2 G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega)G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega)S_\xi(\underset{\sim}{r}_1,\underset{\sim}{r}_2); \tag{2.2.39}$$

for S_A^{bulk} we find

$$S_A^{bulk}(\omega) = \int_{V_s}\int_{V_s} d^3r d^3r' \int_V\int_V d^3r_1 d^3r_2 G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega)G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega)S_\xi(\underset{\sim}{r}_1,\underset{\sim}{r}_2). \tag{2.2.40}$$

These are the sources Green's function forms.

It is well known that Green's functions can be expanded in a biorthogonal series

$$G(\underset{\sim}{r},\underset{\sim}{r}',i\omega) = \sum_k \frac{\phi_k(\underset{\sim}{r})\psi_k^*(\underset{\sim}{r}')}{\lambda_k+i\omega}. \tag{2.2.41}$$

When (2.2.41) is substituted in (2.2.39), the result (2.2.20) is recovered.

2.2.2 Surface Noise

Now we consider the transport equation (2.2.1) with $\xi \equiv 0$ and while we have stochastic boundary conditions (2.2.4); we also assume that ξ and ζ are uncorrelated. (If this is not the case there will be a contribution S_a^{cross}.)

2.2.2(a) Eigenfunction expansion

This method has not been tried due to the complexity of a series expansion in eigenfunctions; however a response method without Green's functions can be derived.

2.2.2(b) Green's function procedure

Again, we make a Fourier analysis of the (homogeneous) Langevin equation and of the stochastic boundary condition (2.2.4)

$$(i\omega_n + \Lambda_{\underset{\sim}{r}})a_n(\underset{\sim}{r}) = 0, \tag{2.2.42}$$

$$ma_n(\underset{\sim}{r}_o) = \zeta_n(\underset{\sim}{r}_o). \tag{2.2.43}$$

The solution of these equations leads to

$$a_n(\underset{\sim}{r}) = \oint d\sigma_o H(\underset{\sim}{r}_1 \underset{\sim}{r}_o, i\omega_n)\zeta_n(\underset{\sim}{r}_o) \tag{2.2.44}$$

$(d\sigma_o = d^2 r_o)$, where H is the surface Green's function. For the spectra we now obtain due to the surface sources

$$S_a^{surf}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = \oint\oint d\sigma_o \; d\sigma_o' H(\underset{\sim}{r},\underset{\sim}{r}_o,i\omega)H(\underset{\sim}{r}',\underset{\sim}{r}_o',-i\omega)S_\zeta(\underset{\sim}{r}_o,\underset{\sim}{r}_o'), \tag{2.2.45}$$

while for S_A^{surf} one obtains

$$S_A^{surf}(\omega) = \int_{V_s}\int_{V_s} d^3r d^3r' \oint_S \oint_S d\sigma \; d\sigma_o H(\underset{\sim}{r},\underset{\sim}{r}_o,i\omega)H(\underset{\sim}{r}',\underset{\sim}{r}_o,-i\omega)S_\zeta(\underset{\sim}{r}_o,\underset{\sim}{r}_o'). \tag{2.2.46}$$

In particular, let $S_\zeta = Z(\underset{\sim}{r}_o)\delta^{(2)}(\underset{\sim}{r}_o-\underset{\sim}{r}_o')$. Then

$$S_A^{surf}(\omega) = \oint_S d^2r_o Z(\underset{\sim}{r}_o)\left|\int_{V_s} d^3r H(\underset{\sim}{r},\underset{\sim}{r}_o,i\omega)\right|^2. \tag{2.2.47}$$

These are the sources Green's function forms. The covariance function associated with the surface noise is

$$\Gamma^{surf}(\underset{\sim}{r},\underset{\sim}{r}') = \int_0^\infty \frac{d\omega}{2\pi}\mathrm{Re}\; S_a^{surf}(\underset{\sim}{r},\underset{\sim}{r}',\omega). \tag{2.2.48}$$

It can also be shown that $\Gamma^{surf}(\underset{\sim}{r},\underset{\sim}{r}')$ satisfies the <u>homogeneous</u> Λ-theorem

$$(\Lambda_{\underset{\sim}{r}}+\Lambda_{\underset{\sim}{r}'})\;\Gamma^{surf}(\underset{\sim}{r},\underset{\sim}{r}') = 0 \qquad (\underset{\sim}{r},\underset{\sim}{r}' \in V). \tag{2.2.49}$$

Total covariance $\Gamma = \Gamma^{bulk}+\Gamma^{surf}$, therefore, satisfies the Λ-theorem which is

$$(\Lambda_{\underset{\sim}{r}}+\Lambda_{\underset{\sim}{r}'})\Gamma(\underset{\sim}{r},\underset{\sim}{r}') = \frac{1}{2}S_\zeta(\underset{\sim}{r},\underset{\sim}{r}') = \Xi(\underset{\sim}{r},\underset{\sim}{r}'), \tag{2.2.50}$$

where Ξ is the correlation strength of the volume sources.

2.3 The Phenomenological Equation and Correlation Methods

The phenomenological equation corresponding to (2.2.1) is

$$L\langle a(\underset{\sim}{r},t)\rangle_{a(\underset{\sim}{r},0)} = (\frac{\partial}{\partial t}+\Lambda_{\underset{\sim}{r}})\langle a(\underset{\sim}{r},t)\rangle_{a(\underset{\sim}{r},0)} = 0, \tag{2.3.1}$$

where $\langle\rangle_{a(\underset{\sim}{r},0)}$ means a conditional ensemble average for systems with fixed fluctuations at time t=0. The boundary conditions corresponding to (2.2.3) or (2.2.4) is

$$m\langle a(\underset{\sim}{r}_b,t)\rangle_{a(\underset{\sim}{r},0)} = 0. \tag{2.3.2}$$

2.3.1 The Volume Noise

Since the surface sources average to zero, it is unlikely that the noise found by correlation method includes surface noise. Moreover, the correlation method does not give the complete bulk noise; in fact, it only gives what is identified as the volume noise in section 2.1. The reason for this is that the covariance function Γ, which figures centrally in this method, only represents the solution of the inhomogeneous Λ-theorem. The surface source and other boundary spectra give covariances which satisfy the homogeneous Λ-theorem.

2.3.1(a) Eigenfunction expansion

The stochastic variable is once more expanded in eigenfunctions $\phi_k(\underset{\sim}{r})$, which satisfy equation (2.3.2), this yields for for equation (2.3.1)

$$\langle\dot{a}_k(t)\rangle_{\{a_k(0)\}} + \lambda_k\langle a_k(t)\rangle_{\{a_k(0)\}} = 0, \tag{2.3.3}$$

which gives

$$\langle a_k(t)\rangle_{\{a_k(0)\}} = \langle a_k(0)\rangle e^{-\lambda_k t} \tag{2.3.3a}$$

and furthermore

$$\Phi_{k\ell}(t) \equiv \langle a_k(t)a_\ell^*(0)\rangle_s = \langle\langle a_k(t)\rangle_{\{a_k(0)\}}a_\ell^*(0)\rangle_s = e^{\lambda_k t}\langle a_k(0)a_\ell^*(0)\rangle_s. \tag{2.3.3b}$$

Finally for the solution one obtains

$$S^{vol}_{a_k a_\ell} = 2\int_0^\infty dt\,[e^{-i\omega t}\langle a_k(t)a^*_\ell(0)\rangle_s + e^{i\omega t}\langle a_k(0)a^*_\ell(t)\rangle]$$

$$= 2\langle a_k a^*_\ell\rangle_s\left(\frac{1}{\lambda_k+i\omega} + \frac{1}{\lambda^*_\ell-i\omega}\right). \tag{2.3.4}$$

We have used here the transposition property for the correlation function $\Phi_{k\ell}(t) = \Phi_{\ell k}(-t)$. With an expression similar to (2.2.19) we find

$$S^{vol}_a(\mathfrak{r},\mathfrak{r}',\omega) = 2\sum_{k\ell}\phi_k(\mathfrak{r})\phi^*_\ell(\mathfrak{r}')\langle a_k a^*_\ell\rangle_s\left(\frac{1}{\lambda_k+i\omega} + \frac{1}{\lambda^*_\ell-i\omega}\right). \tag{2.3.5}$$

Next we consider the expansion for the covariance function

$$\Gamma(\mathfrak{r}_1,\mathfrak{r}_2) \equiv \langle\Delta a(\mathfrak{r}_1,t)\Delta a(\mathfrak{r}_2,t)\rangle$$

$$= \sum_{k\ell}\langle a_k a^*_\ell\rangle_s\phi_k(\mathfrak{r}_1)\phi^*_\ell(\mathfrak{r}_2). \tag{2.3.6}$$

Multiplying with $\psi^*_k(\mathfrak{r}_1)\ \psi_\ell(\mathfrak{r}_2)$ and integrating over all $\mathfrak{r}_1$ and $\mathfrak{r}_2$, one obtains

$$\langle a_{k'}a^*_{\ell'}\rangle = \iint d^3r_1 d^3r_2\Gamma(\mathfrak{r}_1\mathfrak{r}_2)\psi^*_{k'}(\mathfrak{r}_1)\psi_{\ell'}(\mathfrak{r}_2). \tag{2.3.7}$$

Substituting this into equation (2.3.5) we obtain

$$S^{vol}_a(\mathfrak{r},\mathfrak{r}',\omega) = 2\sum_{k\ell}\int_V\int_V d^3r_1 d^3r_2\phi_k(\mathfrak{r})\phi^*_\ell(\mathfrak{r}')\psi^*_k(\mathfrak{r}_1)\psi_\ell(\mathfrak{r}_2)$$

$$\times\left[\frac{1}{\lambda_k+i\omega} + \frac{1}{\lambda^*_\ell-i\omega}\right]\Gamma(\mathfrak{r}_1,\mathfrak{r}_2). \tag{2.3.8}$$

Carrying out the required integration, the result is obtained as

$$S^{vol}_a(\mathfrak{r},\mathfrak{r}',\omega) = 2\sum_k\int_V d^3r''\phi_k(\mathfrak{r})\psi^*_k(\mathfrak{r}'')\left(\frac{1}{\lambda_k+i\omega}\right)\Gamma(\mathfrak{r}'',\mathfrak{r}') + \text{hcj}, \tag{2.3.9}$$

which is the same result as obtained previously via the sources method in equation (2.2.29). For the result of the noise of $A(t) = \int_{V_s} d^3r\, a(\mathfrak{r},t)$, using this relation one obtains (2.2.33) or (2.2.34).

2.3.1(b) Green's function procedure

For this procedure we need the solution of equation (2.3.1); for this purpose we Fourier-Laplace transform equation (2.3.1), i.e. let

$$\hat{a}(\mathfrak{r},i\omega) = \int_0^\infty e^{-i\omega t}\langle a(\mathfrak{r},t)\rangle_{a(\mathfrak{r},0)}\, dt. \tag{2.3.10}$$

Then equation (2.3.1) yields

$$(i\omega+\Lambda_r)\hat{a}(\mathfrak{r},i\omega) = a(\mathfrak{r},0). \tag{2.3.11}$$

The Green's function is given by

$$(i\omega+\Lambda_{\mathfrak{r}})G(\mathfrak{r},\mathfrak{r}',i\omega) = \delta(\mathfrak{r}-\mathfrak{r}'), \tag{2.3.12}$$

with the boundary condition $mG(\mathfrak{r}_o,\mathfrak{r}',i\omega) = 0$ corresponding to equation (2.3.2). By the standard procedure we find the solution as

$$\hat{a}(\mathfrak{r},i\omega) = \int_V d^3r''G(\mathfrak{r},\mathfrak{r}'',i\omega)a(\mathfrak{r}'',0). \tag{2.3.13}$$

Therefore, by inversion of the Fourier-Laplace transform

$$\langle a(\mathfrak{r},t)\rangle_{a(\mathfrak{r},0)} = -\frac{1}{2\pi}\oint_C d\bar{\omega}e^{i\bar{\omega}t}\int_V d^3r''G(\mathfrak{r},\mathfrak{r}'',i\bar{\omega})a(\mathfrak{r}'',0) \tag{2.3.14}$$

where ω is a complex frequency and C is contour encircling all poles. For the correlation function we have ($t \geq 0$)

$$\Phi(\mathfrak{r},\mathfrak{r}',t) \equiv \langle a(\mathfrak{r},t)a(\mathfrak{r}',0)\rangle_s = \langle\langle a(\mathfrak{r},t)\rangle_{a(\mathfrak{r}',0)} a(\underset{\sim}{r}',0)\rangle_s$$

$$= -\frac{1}{2\pi}\oint_C d\bar{\omega}e^{i\bar{\omega}t}\int_V d^3r''G(\mathfrak{r},\mathfrak{r}'',i\bar{\omega})\Gamma^{bulk}(\mathfrak{r}',\mathfrak{r}''). \qquad (2.3.15)$$

For negative time (-t) we use the transposition property $\phi(\mathfrak{r},\mathfrak{r}',-t) = \phi(\mathfrak{r}',\mathfrak{r},t)$. For the spectrum the Wiener-Khintchine theorem then yields,

$$S_a^{vol}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = -\frac{1}{\pi}\int_0^\infty dt\oint_C d\bar{\omega}e^{i(\bar{\omega}-\omega)t}\int_V d^3r''G(\mathfrak{r},\mathfrak{r}'',i\bar{\omega})\Gamma^{bulk}(r',r'') + hcj. \qquad (2.3.16)$$

Interchanging the time and contour integral we have

$$\int_C d\bar{\omega}\int_0^\infty dt\, e^{i(\bar{\omega}-\omega)t}G(\mathfrak{r},\mathfrak{r}'',i\bar{\omega}) = -\oint_C d\bar{\omega}\left[\frac{1}{i(\bar{\omega}-\omega)} + \pi\delta(\bar{\omega}-\omega)\right]G(\mathfrak{r},\mathfrak{r}'',i\bar{\omega})$$

$$= -2\pi G(\mathfrak{r},\mathfrak{r}'',i\omega+0)\text{(by Cauchy's theorem)} \qquad (2.3.17)$$

So now substituting the integral in equation (2.3.16) we get the result

$$S_a^{vol}(\mathfrak{r},\mathfrak{r}',\omega) = 2\int_V d^3r''G(\mathfrak{r},\mathfrak{r}'',i\omega)\Gamma^{bulk}(\mathfrak{r}',\mathfrak{r}'') + hcj. \qquad (2.3.18)$$

This is van-Vliet-Fassett form. The result is in full accord with equation (2.3.9) if we use the series form of the Green's function, equation (2.2.41).

2.3.2 The Boundary Noise

The spectrum found by the above procedure does not include surface noise. However, it also does not comprise all the bulk noise,

which is to some extent surprising. The reason is that the transport equation is not Markovian in the usual sense. The fact that the dimensionality is infinite means that the usual Markov property

$$W_2(a(\underset{\sim}{r},t),a'(\underset{\sim}{r}',t')) = P(a(\underset{\sim}{r},t)\,|a'(\underset{\sim}{r}',t'))W_1(a'(\underset{\sim}{r}',t')), \qquad (2.3.19)$$

which is the basis for the evaluation of the correlation function in (2.3.15) must be doubted. In other words, the process cannot be Markovian unless boundary behaviour is also specified.

The total noise can be broken up in two ways. The first way is

$$S^{total} = S^{bulk} + S^{surface}\ (+S^{cross}). \qquad (2.3.20)$$

This breaking up is according to the considerations of section 2, the terms representing effects due to the bulk sources $\xi(\underset{\sim}{r},t)$, the surface sources $\zeta(\underset{\sim}{r}_0,t)$ and due to their possible correlation. In the second way we evaluate volume noise according to section (2.3.1) based on (2.3.15). The remaining part to be evaluated in this section is called $S^{boundary}$. Thus

$$S^{total} = S^{volume} + S^{boundary}(+S^{cross}). \qquad (2.3.21)$$

To obtain $S^{boundary}$ we start with the response expression (2.2.39). In the RHS we substitute for S_ξ the Λ-theorem, and obtain

$$S_a^{bulk}(\underset{\sim}{r},\underset{\sim}{r}',\omega) = 2\int_V\int_V d^3r_1 d^3r_2 G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega)G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega)(\Lambda_{\underset{\sim}{r}_1}+\Lambda_{\underset{\sim}{r}_2})\Gamma(\underset{\sim}{r}_1,\underset{\sim}{r}_2). \qquad (2.3.22)$$

We indicate by $\hat{\Gamma}(\underset{\sim}{r}_1,\underset{\sim}{r}_2)$ here any (interior domain) particular solution of the Λ theorem, which may differ from physical covariance function by a solution of the homogeneous Λ theorem. The RHS of (2.3.22) is rewritten as

$$\begin{aligned}
\text{RHS} &= 2\int_V d^3r_1\, G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega)\int_V d^3r_2\, G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega)(-i\omega+\Lambda_{\underset{\sim}{r}_2})\hat{\Gamma}(\underset{\sim}{r}_1,\underset{\sim}{r}_2) \\
&\quad +2\int_V d^3r_2\, G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega)\int_V d^3r_1\, G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega)(i\omega+\Lambda_{\underset{\sim}{r}_1})\hat{\Gamma}(\underset{\sim}{r}_1,\underset{\sim}{r}_2) \\
&= 2\int_V d^3r_1\, G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega)\{\int_V d^3r_2\hat{\Gamma}(\underset{\sim}{r}_1,\underset{\sim}{r}_2)L^{\dagger}_{\underset{\sim}{r}_2}\, G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega) \\
&\quad +\oint_S d^2r_0 C[\hat{\Gamma}(\underset{\sim}{r}_1,\underset{\sim}{r}_0),G(\underset{\sim}{r}',\underset{\sim}{r}_0,-i\omega)]\} \\
&\quad +2\int_V d^3r_2 G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega)\{\int_V d^3r_1\hat{\Gamma}(\underset{\sim}{r}_1,\underset{\sim}{r}_2)L^{\dagger *}_{\underset{\sim}{r}_1}\, G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega) \\
&\quad +\oint_S d^2r_0 C[\hat{\Gamma}(\underset{\sim}{r}_0,\underset{\sim}{r}_2),G(\underset{\sim}{r},\underset{\sim}{r}_0,i\omega)]\},
\end{aligned} \tag{2.3.23}$$

where we have used Green's theorem in the form

$$(L\, f,g)-(f,\, L^{\dagger}g) = (\Lambda f,g)-(f,\Lambda^{\dagger}g) = \oint d\sigma\; C[f,g^*]. \tag{2.3.24}$$

According to the definitional equation for the Green's function

$$L^{\dagger}_{\underset{\sim}{r}_2}\, G(\underset{\sim}{r}',\underset{\sim}{r}_2,-i\omega) = L^{\dagger}_{\underset{\sim}{r}_2}\, G^{\dagger}(\underset{\sim}{r}_2,\underset{\sim}{r}',i\omega) = \delta(\underset{\sim}{r}_2-\underset{\sim}{r}'), \tag{2.3.25a}$$

$$L^{\dagger *}_{\underset{\sim}{r}_1}\, G(\underset{\sim}{r},\underset{\sim}{r}_1,i\omega) = L^{\dagger *}_{\underset{\sim}{r}_1}\, G^*(\underset{\sim}{r}_1,\underset{\sim}{r},i\omega) = \delta(\underset{\sim}{r}_1-\underset{\sim}{r}). \tag{2.3.25b}$$

Carrying out the delta function integrations we are left with

$$\begin{aligned}
S_a^{\text{bulk}}(\underset{\sim}{r},\underset{\sim}{r}',\omega) &= 2\int_V d^3r''\, G(\underset{\sim}{r},\underset{\sim}{r}'',i\omega)\hat{\Gamma}(\underset{\sim}{r}'',\underset{\sim}{r}') + \text{hcj} \\
&\quad +2\int d^3r''\, G(\underset{\sim}{r},\underset{\sim}{r}'',i\omega)\oint_S d^2r_0\; C[\hat{\Gamma}(\underset{\sim}{r}'',\underset{\sim}{r}_0),G(\underset{\sim}{r}',\underset{\sim}{r}_0,-i\omega)] + \text{hcj}.
\end{aligned} \tag{2.3.26}$$

The first two terms comprise S_a^{vol}. The last two terms, together with S_a^{surf} comprise $S_a^{boundary}$. The Green's function form for S_a^{surf} was given in (2.2.45). We then have

$$S_a^{bound}(\mathfrak{r},\mathfrak{r}', \omega) = 2 \int_V d^3r'' \, G(\mathfrak{r},\mathfrak{r}'',i\omega) \oint_S d^2r_0 \, C[\hat{\Gamma}(\mathfrak{r}'',\mathfrak{r}_0),G(\mathfrak{r}',\mathfrak{r}_0,-i\omega)]$$

$$+ \text{hcj} + \int_{V_s} \int_{V_s} d^3r \, d^3r' \oint_S \oint_S d^2r_0 \, d^2r_0' \, H(\mathfrak{r},\mathfrak{r}_0,i\omega)H(\mathfrak{r}',\mathfrak{r}_0',-i\omega). \quad (2.3.27)$$

We note that all terms in (2.3.27) are quadratic in the Green's function. Thus the division of S into S^{vol} and S^{bound} separates the spectra in expressions linear and quadratic in G. The merit of this separation is that in many instances all terms of $S^{boundary}$ are zero for homogeneous systems and delta covariance function. For inhomogeneous systems such as transitors, the boundary terms are of great importance, however;when $S^{boundary} = 0$, the evaluation of the noise by S^{vol} as given in section (2.3.1(b)) is by far the fastest procedure.

In equation (2.3.26) we used a particular solution $\hat{\Gamma}$. Another choice of this solution, i.e., $\hat{\Gamma}'$, leads to another division of S^{vol} and $S^{boundary}$. However, the total result (2.3.26) is unique. Usually one chooses $\hat{\Gamma}$ such that the spectrum $S^{boundary}$ vanishes, if possible. If on physical grounds it can be argued that there is no boundary spectrum contribution, then this choice of $\hat{\Gamma}$ is clearly the physical covariance function.

So far we have discussed the mathematical methods to obtain transport noise spectra. In the next chapter we shall discuss and briefly explain the different physical noise sources, their spectra and importance in calculating overall noise spectra of the physical system.

CHAPTER 3
PHYSICAL NOISE SOURCES

3.1 Introduction

We have seen in the previous chapter that a calculation of noise spectra with the Langevin or the sources method requires that the spectra of the Langevin noise source must be known. There are various physical noise sources with known spectra. Furthermore, we have also seen that there are two different types of noise, viz. first volume or bulk noise and second surface or boundary noise stemming from volume noise sources and surface noise sources, respectively. In this chapter we shall try to explain both types of noise sources as we encounter them in physical systems.

3.2 Volume Noise Sources[15]

3.2.1 Particle Diffusion Noise Source

To explain the diffusion noise source let us assume that the fluctuations in number of particles (electrons or holes) are governed by the diffusion equation, say for electrons

$$\frac{\partial}{\partial t}(\Delta n) + D\nabla^2(\Delta n) = 0 \qquad (3.2.1)$$

where Δn is the fluctuation in density of electrons.

The Langevin noise source representation of this diffusion equation in the form of the standard diffusion noise source $\xi(\underset{\sim}{r},t)$ is given by

$$\frac{\partial}{\partial t}(\Delta n) + D\nabla^2(\Delta n) = \xi(\underset{\sim}{r},t). \tag{3.2.2}$$

The source ξ follows from the stochastic current as follows. For $(\Delta\underset{\sim}{J})_{diff}$ we have the randomness

$$(\Delta\underset{\sim}{J})_{diff} = -qD\underset{\sim}{\nabla}(\Delta n) + q\underset{\sim}{\eta}(\underset{\sim}{r},t). \tag{3.2.3}$$

On a microscopic scale, $\underset{\sim}{\eta}(\underset{\sim}{r},t)$ represents the thermal scattering or Brownian motion velocity fluctuations of the carriers. This η source has a spectrum[15]

$$\underset{\underset{\sim}{\eta}}{\underline{\underline{S}}} = 4D\langle\eta(\underset{\sim}{r})\rangle\delta(\underset{\sim}{r}-\underset{\sim}{r}')\underline{\underline{I}} \tag{3.2.4}$$

where $\underline{\underline{I}}$ is the unit tensor.

We also write $\langle n(\underset{\sim}{r})\rangle \equiv n_0(\underset{\sim}{r})$. The conservation theorem for charge reads

$$\frac{\partial\Delta n}{\partial t} - \frac{1}{q}\underset{\sim}{\nabla}\cdot(\Delta\underset{\sim}{J})_{diff} = 0. \tag{3.2.4a}$$

Substituting (3.2.3) we obtain

$$\frac{\partial\Delta n}{\partial t} + D\nabla^2(\Delta n) = \underset{\sim}{\nabla}\cdot\underset{\sim}{\eta}(\underset{\sim}{r},t). \tag{3.2.5}$$

Comparing (3.2.5) with (3.2.2), we note

$$\xi(\underset{\sim}{r},t) = \underset{\sim}{\nabla}\cdot\underset{\sim}{\eta}(\underset{\sim}{r},t).$$

Hence for the spectrum

$$S_\xi(\mathfrak{x},\mathfrak{x}') = \underset{\sim}{\nabla}_{\mathfrak{x}}\underset{\sim}{\nabla}_{\mathfrak{x}'} : \underset{=}{S}_\eta(\mathfrak{x},\mathfrak{x}') \tag{3.2.6a}$$

$$= 4D\underset{\sim}{\nabla}_{\mathfrak{x}}\cdot\underset{\sim}{\nabla}_{\mathfrak{x}'}\{n_0(\mathfrak{x})\delta(\mathfrak{x}-\mathfrak{x}')\}. \tag{3.2.6b}$$

This diffusion noise source is the proper noise source of statistical mechanics. As we can see by the Λ-theorem, due to van Vliet[8], which connects Langevin's noise source spectra to the covariance function,

$$(\Lambda_{\mathfrak{x}}+\Lambda_{\mathfrak{x}'})\Gamma(\mathfrak{x},\mathfrak{x}') = \frac{1}{2}S_\xi(\mathfrak{x},\mathfrak{x}') \tag{3.2.7}$$

the standard diffusion noise source gives for the covariance function a delta function:

$$\Gamma(\mathfrak{x},\mathfrak{x}') = n_0(\underset{\sim}{r})\delta(\underset{\sim}{r}-\underset{\sim}{r}'). \tag{3.2.8}$$

All the previous calculations of diffusion noise by McFarlane[16], Burgess[17], Richardson[18], van Vliet and Fassett[9], Lax and Mengert[3], are based on the assumption that the spatial covariance function is a delta function given as constant $\times\ \delta(\mathfrak{x}-\mathfrak{x}')$. This result can also be proven from first principles, see ref.9, page 329.

3.2.2 Heat Diffusion Noise Source

The heat diffusion noise source can be explained by considering the relation between the heat conductivity σ and the heat current $\underset{\sim}{K}$, which is

$$\underset{\sim}{K} = -\sigma\underset{\sim}{\nabla}T + \underset{\sim}{\eta}(\mathfrak{x},t) \tag{3.2.9}$$

where $\underset{\sim}{\eta}$ is a Langevin source. Heat conservation requires that

$$cd \frac{\partial T}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{K} = 0 \tag{3.2.10}$$

where c is specific heat per gram, and d is the density. We can find the stochastic heat diffusion equation by substituting (3.2.8) into (3.2.9) and by using $T = \langle T \rangle + \Delta T$,

$$\frac{\partial \Delta T}{\partial t} - \alpha \nabla^2 (\Delta T) = \frac{-\underset{\sim}{\nabla} \cdot \underset{\sim}{\eta}(\underset{\sim}{r},t)}{cd} \equiv \xi(\underset{\sim}{r},t) \tag{3.2.11}$$

where $\alpha = \sigma/cd$ is the thermal diffusion constant. The correlation strength or white noise spectrum, of the source $\underset{\sim}{\eta}$ has been found by van Vliet[8] using the Boltzmann transport equation, to be

$$\underset{=}{S}_{\eta}(\underset{\sim}{r},\underset{\sim}{r}') = 4\underset{=}{I} k\, T(\underset{\sim}{r})^{2} \sigma(\underset{\sim}{r}) \delta(\underset{\sim}{r}-\underset{\sim}{r}'), \tag{3.2.12}$$

where $T(\underset{\sim}{r})$ is the time or ensemble averaged temperature at position $\underset{\sim}{r}$ and $\hat{\underset{=}{I}}$ is the unit tensor. Clearly

$$S_{\xi}(\underset{\sim}{r},\underset{\sim}{r}') = (4k/c^2d^2)\, \underset{\sim}{\nabla}_{\underset{\sim}{r}} \cdot \underset{\sim}{\nabla}_{\underset{\sim}{r}'} \{ T(\underset{\sim}{r})^{2} \sigma(\underset{\sim}{r}) \delta(\underset{\sim}{r}-\underset{\sim}{r}') \} \tag{3.2.13}$$

which is the spectrum of the standard heat diffusion noise source.[14]

3.2.3 Voss and Clarke's P-Source[19]

Voss and Clarke introduced another noise source, which has a delta function type correlation function and a white spectrum given by

$$S_{\xi_{V-Cl}}(\underset{\sim}{r},\underset{\sim}{r}') = P_0^2 \delta(\underset{\sim}{r}-\underset{\sim}{r}') \tag{3.2.14}$$

where P_0 is a constant.

As we have seen previously noise sources cannot be chosen ad hoc due to limitations imposed by the Λ-theorem and statistical mechanics. Using the Λ-theorem the three dimensional spatial covariance function corresponding to the P noise source is found to be

$$\Gamma(\mathfrak{r},\mathfrak{r}') = \frac{1}{16\pi D} \frac{P_0^2}{|(\underset{\sim}{r}-\underset{\sim}{r}')|} . \tag{3.2.15}$$

Such a long-range correlation function is highly unphysical.

3.2.4 Particle Volume Generation-Recombination Source

Usually noise due to generation and recombination of carriers results in fluctuations in the resistance of the specimen under consideration. The noise can be described by the fluctuating number N of the carriers, say for electrons; then the equation governing the fluctuations in N is

$$\frac{dN}{dt} = g(N) - r(N) + \Delta g(t) - \Delta r(t) \tag{3.2.16}$$

where $g(N)$ and $r(N)$ are generation and recombination rates and $\Delta g(t)$ and $\Delta r(t)$ are the randomness in these rates.

Substituting $N = N_0+\Delta N$, where N_0 is the equilibrium number of carriers and neglecting higher-order terms in ΔN yields the equilibrium condition $g(N_0) = r(N_0)$ and the linearized Langevin equation, given by

$$\frac{d\Delta N}{dt} = -\frac{\Delta N}{\tau} + \Delta g(t) - \Delta r(t) \equiv -\frac{\Delta N}{\tau} + \xi_{gr}(t) \tag{3.2.17}$$

where

$$\frac{1}{\tau} = \left(\frac{dr}{dN} - \frac{dg}{dN}\right)\Big|_{N_0} .$$

The Langevin source $\xi_{gr}(t)$ can be obtained by assigning shot noise to the random transitions Δg and Δr:

$$S_{\Delta g} = 2g(N_0) \equiv 2g_0 \tag{3.2.18}$$

$$S_{\Delta r} = 2r(N_0) \equiv 2r_0 = 2g_0 \tag{3.2.19}$$

$$S_{\Delta g,\Delta r} = 0 \quad \text{(no correlation)}.$$

Hence

$$S_{\xi_{gr}} = S_{\Delta g} + S_{\Delta r} - 2S_{\Delta g,\Delta r} = 4g_0. \tag{3.2.20}$$

Making a Fourier analysis of (3.2.17) one finds for the spectral connection

$$S_{\Delta N}(\omega) = \frac{S_{\xi}\tau^2}{(1+\omega^2\tau^2)} = \frac{4g_0\tau^2}{1+\omega^2\tau^2}. \tag{3.2.21}$$

Integrating over all frequencies, we find

$$\langle \Delta N^2 \rangle = \frac{1}{2\pi}\int_0^\infty S_{\Delta N}(\omega)\, d\omega = g_0\tau = \frac{g_0}{[r'-g']_{N_0}} \tag{3.2.22a}$$

which is Burgess'[20,21] g-r-theorem.

In transport systems the spatial dependence must be added. Thus, for the density $n(\underset{\sim}{r},t)$ subject to diffusion and generation recombination processes we have

$$\frac{\partial \Delta n(r,t)}{\partial t} + D\nabla^2\, \Delta n(\underset{\sim}{r},t) + \frac{\Delta n(\underset{\sim}{r},t)}{\tau} = \xi_{diff}(\underset{\sim}{r},t) + \xi_{gr}(\underset{\sim}{r},t).$$

The g-r noise source now becomes

$$S_{\xi_{gr}}(\underset{\sim}{r},\underset{\sim}{r}') = 4g_0\delta(\underset{\sim}{r}-\underset{\sim}{r}'). \tag{3.2.22b}$$

3.3 Surface Noise Sources

3.3.1 Particle Surface Generation-Recombination Source

Just like generation-recombination processes in the bulk give rise to a bulk Langevin noise source, so generation-recombination processes at the surface can be represented by a surface noise source; this latter source causes surface transport noise, at the surface of the sample.

For example, for the case of non-zero surface recombination velocity, the stochastic boundary condition can be written as

$$D_n \nabla_0 (\Delta n) + S\Delta n = \zeta(\underset{\sim}{r}_0, t) \tag{3.3.1}$$

where D_n is the diffusion constant for electrons $\underset{\sim}{\nabla}_0$ is the surface concentration gradient and $\zeta(\underset{\sim}{r}_0, t)$ is the surface stochastic Langevin noise source. For the threedimensional case this surface noise source has a spectrum

$$S_\zeta = 4M\alpha Sn(\underset{\sim}{r}_0)\delta^{(2)}(\underset{\sim}{r}-\underset{\sim}{r}_0') \tag{3.3.2}$$

and for the one dimensional case

$$S_\zeta = 4M\alpha Sn(X_0)/A$$

where M is the modulation factor ≥ 1, S is the surface recombination velocity, A is the area of surface, $\alpha = \langle \Delta N^2 \rangle / N_0$ and $\delta^{(2)}(\underset{\sim}{r}_0 - \underset{\sim}{r}_0')$ is the two dimensional delta function.

3.3.2 Heat Transfer Source

The heat transfer source is a stochastic source due to absorption and emission of blackbody radiation at the surface of a body. For example, this occurs in the case of a thin film supported by a substrate with the front face of the body being blackened, so that it sees the blackbody radiation field. The boundary condition in this case becomes

$$\sigma(0) \left.\frac{\partial \Delta T}{\partial x}\right|_{x=0} - g\Delta T(0) = \zeta(t). \tag{3.3.3}$$

Here $\zeta(t)$ is the stochastic heat transfer source. The spectrum of this source is given by the generalized Nyquist's theorem

$$S_\zeta = 4kT_0^2 g = 16\sigma_B k T_0^5 \tag{3.3.4}$$

where $g = 4\sigma_B T_0^3$ is the heat transfer conductance, T_0 is the temperature of the environment and the mean temperature of the body, and $\sigma_B = 2\pi^5 k^4/15h^3c^2$ is the Stefan-Boltzmann constant.

PART B
NOISE DUE TO VOLUME NOISE SOURCES

CHAPTER 4
SYMMETRICAL EMBEDDED BODIES

4.1 Introduction

In this chapter we shall study the noise spectra resulting from symmetrical geometries which are part of an infinite expanse i.e. the "embedded" case. We have already seen in a previous chapter that there are two volume noise sources, namely the standard diffusion noise source and Voss and Clarke's 'P' source. Further we have also seen that the 'P' source is highly unphysical. In this chapter we consider only noise stemming from the diffusion noise source.

4.2 Survey of Older Work

In this section we briefly summarize the results of previous work for obtaining noise spectra from symmetrical embedded bodies like the linear bar, the cylinder and the sphere, prior to the discovery of the Green's function method.

MacFarlane[16] in 1950, studied the problem of the power density spectra of noise current resulting from contact noise in semiconductors or from emission noise in field emitter diodes. In these theories low frequency noise is attributed to the random movement of absorbed ions on the contact surface between two granules of a carbon resistor or on the emission tip of a field emitter tube. The current through

the tube or the emission of electrons is assumed to take place only at localized patches on the surface and the absorbed ions are assumed to give rise to a dipole layer, which modulates the work function in such a way that ΔW is linearly related to the concentration of ions in that patch. This concentration fluctuation is due to diffusion of ions over the surface of which the conducting patch is a small part.

MacFarlane found, by first calculating the autocorrelation function and then using the Wiener-Khintchine theorem, that the spectral power density of the noise current is given by

$$R(\omega) \propto \int_0^\infty \{1-\exp(-x^{1/4})[I_0(x^{-1/4})+I_1(x^{-1/4})]\}\cos(px)\, dx \qquad (4.2.1)$$

where $p = \omega\tau_0$, $\tau_0 = (2r)^2/D$, r is the radius of the patch, and D is the diffusion constant of ions, and I's are the modified Bessel functions. In the limiting cases it was shown that, for a circular patch, (using probability theory results[22].)

$$R(\omega) \propto 0.2345\, p^{-3/4}, \quad p \ll 1 \qquad (4.2.2a)$$

and

$$R(\omega) \propto 0.1466\, p^{-9/8}, \quad p \gg 1 \qquad (4.2.2b)$$

while for a long thin strip

$$R(\omega) \propto (2p)^{-1/2}, \qquad p \ll 1 \qquad (4.2.3a)$$

and

$$R(\omega) \propto (2p)^{-3/2}, \qquad p \gg 1. \qquad (4.2.3b)$$

For the long thin strip this shows the occurrence of the 3/2 power law at high frequencies. This universal law was violated by the result (4.2.2b); Burgess[17] showed MacFarlane's result to be in error; when properly integrated, a 3/2 power law at high frequencies also result for the circular patch.

4.3 Previous Work Using the Green's Function Method

4.3.1 Linear Bar

Van Vliet-Fassett[9] considered similar problems, as discussed above, using the Green's function procedure. For the one-dimensional diffusion equation

$$\frac{\partial \Delta p}{\partial t} = D \frac{\partial^2 \Delta p}{\partial x^2} , \tag{4.3.1}$$

where Δp is a fluctuation variable and D is a diffusion constant, the Green's function is found by standard procedure to be

$$G(x,i\omega,x') = \frac{i}{2u_0 D} \exp(iu_0 |x-x'|) \tag{4.3.2}$$

where $u_0 = \sqrt{\omega/D}\, e^{3i\pi/4}$.

The spectrum for the fluctuations ΔP, where P is the total number of particles in a segment L is found to be

$$S_{\Delta P}(\omega) = \frac{\langle \Delta P^2 \rangle}{D\theta^3} \left(1-e^{-\theta}(\cos\theta+\sin\theta)\right) \tag{4.3.3}$$

where $\theta = L(\frac{\omega}{2D})^{1/2}$, in agreement with Burgess' solution[17] of the integral given by MacFarlane[16]. The low frequency spectrum goes as $\omega^{-1/2}$ and the high frequency asymptote is again $\omega^{-3/2}$. The latter behavior is characteristic for all diffusion spectra.

4.3.2 Cylinder or Circular Patch

Van Vliet and Chenette[23] studied the noise spectra resulting from diffusion processes in cylindrical geometries. The spectral densities are derived from the relevant Green's functions. Also extensions to MacFarlane's[16] spectrum for the circular patch were obtained.

The noise spectrum for the fluctuating variable X, where X is the total number of particles in the cylindrical volume V_s, in terms of the relevant Green's function, is given by

$$S_X(\omega) = \frac{4\mathrm{var}X}{V_s} \int_{D_s} \int_{D_s} G(\underset{\sim}{r}, i\omega, \underset{\sim}{r}')dv\, dv' \tag{4.3.4}$$

where V_s is the volume of domain D_s (cylinder or circle), which is part of an expanse $V >> V_s$.

The Green's function was found to be the Hankel function of the first kind of order zero; the spectrum is given by

$$S_X(\omega) = \frac{8\pi(\mathrm{var}X)a^2}{D} \mathrm{Re}\, \frac{1}{(\varkappa a)^2} [-\frac{1}{2\pi} + \frac{i}{2}H_1^{(1)}(\varkappa a)J_1(\varkappa a)], \tag{4.3.5}$$

where H and J are Hankel and Bessel functions, a is the radius of the cylinder, D is the diffusion constant and $\varkappa = i\sqrt{\frac{i\omega+\alpha}{D}}$.

It was found that for low frequencies the spectrum has logarithmic behavior and at high frequencies the $\omega^{-3/2}$ behavior takes over.

4.3.3 Sphere

The problem of three dimensional diffusion has been solved by van Vliet and Fassett[9]. Using the method of reciprocity, the spectrum was found to be (for fluctuating variable ΔP, with $\overline{\Delta P^2} = P_0$),

$$G_{\Delta P}(\omega) = \frac{16\pi P_0 R_0^5}{D\gamma^5} \{\gamma^2 - 2 + e^{-\gamma}[\gamma^2(\cos\gamma + \sin\gamma) + 4\gamma\cos\gamma + 2(\cos\gamma - \sin\gamma)]\}, \tag{4.3.6}$$

where

P_0 - average number of particles in the sphere

R_0 - radius of sphere

D - diffusion constant

and

$$\gamma = 2R_0\sqrt{(\omega/2D)} .$$

At low and high frequencies the spectrum was found to have the limiting values:

$$G_{\Delta P}(\omega)_{\omega\to 0} = \frac{32\pi P_0 R_0^5}{15D} \tag{4.3.7a}$$

$$G_{\Delta P}(\omega)_{\omega\to\infty} = \frac{P_0(2D)^{1/2} 4\pi R_0^2}{\omega^{3/2}} . \tag{4.3.7b}$$

Again, at high frequencies the spectrum follows the 'universal' 3/2 power law.

4.4 Present Work: Carrier Fluctuations in a Sphere Due to Diffusion and Generation-Recombination

In this section we investigate the noise spectra resulting from fluctuations in the carriers in a sphere, due to diffusion across the boundaries and due to sinks or sources inside the sphere. This problem has not been solved hitherto.

We consider a sphere of radius 'a' (volume V_s) which is part of a very large expanse. The fluctuating variable is x, the concentration of particles in the spherical volume under consideration; the total number of particles in the sphere is X. The boundary conditions on the surface D_s of the sphere are 'fictitious' i.e. x and its derivatives are continuous across the boundary.

We consider the equation

$$\frac{\partial x}{\partial t} + \alpha x - D\nabla^2 x = 0 \tag{4.4.1}$$

where $\alpha^{-1} = \tau$ is the volume lifetime and D is a diffusion constant. The Green's function $g(\underset{\sim}{r},t|\underset{\sim}{r}',t')$ satisfies the equation

$$\frac{\partial g}{\partial t}(\underset{\sim}{r},t|\underset{\sim}{r}',t') + \alpha g(\underset{\sim}{r},t|\underset{\sim}{r}',t') - D\nabla^2 g(\underset{\sim}{r},t|\underset{\sim}{r}',t') = \delta(t-t')\delta(\underset{\sim}{r}-\underset{\sim}{r}'). \tag{4.4.2}$$

Assuming $t' = 0^+$ as initial time, with $g(r,0|r',0)=0$, and taking the Laplace transform of equation (4.3.2) we obtain,

$$sG(\underset{\sim}{r},s,\underset{\sim}{r}') + \alpha G(\underset{\sim}{r},s,\underset{\sim}{r}') - D\nabla^2 G(\underset{\sim}{r},s,\underset{\sim}{r}') = \delta(\underset{\sim}{r}-\underset{\sim}{r}'). \tag{4.4.3}$$

This can be written as

$$-\frac{sG}{D}(\underset{\sim}{r},s,\underset{\sim}{r}') - \frac{\alpha}{D}G(\underset{\sim}{r},s,\underset{\sim}{r}') + \nabla^2 G(\underset{\sim}{r},s,\underset{\sim}{r}') = -\frac{1}{D}\delta(\underset{\sim}{r}-\underset{\sim}{r}') \tag{4.4.4}$$

or

$$(\varkappa^2+\nabla^2)G(\underset{\sim}{r},s,\underset{\sim}{r}') = -\frac{1}{D}\delta(\underset{\sim}{r}-\underset{\sim}{r}') \tag{4.4.5}$$

where

$$\varkappa = i\sqrt{\frac{\alpha+s}{D}} \equiv i\sqrt{\frac{\alpha+i\omega}{D}} . \tag{4.4.6}$$

Since the boundary conditions on D_s are continuity of x and its derivative and the domain D is assumed to be very large, the ultimate boundary condition on the boundary of D is that G should behave properly at infinity, i.e. go sufficiently fast to zero.

Adopting a spherical coordinate system we have

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) . \tag{4.4.7}$$

Following Morse and Feshbach[24], page 808, one can show that for the infinite domain the Green's functions in spherical coordinates only depends on $|\underset{\sim}{r}-\underset{\sim}{r}'| = R$. So, evaluating ∇^2 in spherical coordinates with $R = 0$ as origin, we find

$$\nabla^2 = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial}{\partial R})$$

$$= \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} . \tag{4.4.8}$$

Also we can write

$$G(\underset{\sim}{r},s,\underset{\sim}{r}') = G_\varkappa(R) . \tag{4.4.9}$$

So, the Green's function equation (4.4.5) becomes

$$\frac{\partial^2 G_\varkappa(R)}{\partial R^2} + \frac{2}{R}\frac{\partial G_\varkappa(R)}{\partial R} + \varkappa^2 G_\varkappa(R) = -\frac{1}{D}\frac{\delta(R)}{4\pi R^2} . \tag{4.4.10}$$

Following Morse and Feshbach[26], page 809, we need a solution for the above equation which goes as

$$\frac{1}{R} \quad \text{for} \quad R \to 0. \tag{4.4.11}$$

Therefore we require the solution to satisfy the following conditions

i) $$G_{\varkappa}(R) \to \frac{1}{R} \quad \text{as} \quad R \to 0. \tag{4.4.12a}$$

In addition, the noise spectrum for high frequencies $\omega \to \infty$ should go to zero, or

ii) $$\lim_{\varkappa \to -\infty + i\infty} G_{\varkappa}(R) = 0. \tag{4.4.12b}$$

The only solution for equation (4.4.10), satisfying the above two conditions is

$$G_{\varkappa}(R) = \frac{e^{i\varkappa R}}{4\pi RD} . \tag{4.4.13}$$

The proof of the solution is as follows.

$$\frac{\partial}{\partial R} G_{\varkappa}(R) = \frac{\pm i\varkappa e^{\pm i\varkappa R}}{4\pi RD} + \frac{e^{\pm i\varkappa R}}{D} \frac{\partial}{\partial R} \frac{1}{4\pi R} , \tag{4.4.14}$$

$$\frac{\partial^2}{\partial R^2} G_{\varkappa}(R) = -\varkappa^2 G \pm \frac{2i\varkappa e^{\pm i\varkappa R}}{D} \frac{\partial}{\partial R} \frac{1}{4\pi R} + \frac{e^{\pm i\varkappa R}}{D} \frac{\partial^2}{\partial R^2} \frac{1}{4\pi R} \tag{4.4.15}$$

$$\frac{2}{R} \frac{\partial}{\partial R} G_{\varkappa}(R) = \pm \frac{2i\varkappa e^{\pm i\varkappa R}}{4\pi R^2 D} + \frac{2e^{\pm i\varkappa R}}{DR} \frac{\partial}{\partial R} \frac{1}{4\pi R} . \tag{4.4.16}$$

The second term of (4.4.15) can be shown to not contain a delta function; it cancels the first term of (4.4.16). The third term of (4.4.15) and the second term of (4.4.16) are combined to yield

$$\frac{e^{\pm i\varkappa R}}{D} \left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R}\right) \frac{1}{4\pi R} = \frac{e^{\pm i\varkappa R}}{D} \nabla^2 \frac{1}{4\pi R} . \tag{4.4.17}$$

For the latter we have

$$\frac{e^{\pm i\varkappa R}}{D}\nabla^2\frac{1}{4\pi R} = -\frac{e^{\pm i\varkappa R}}{D}\delta(\mathfrak{r}-\mathfrak{r}') = -\frac{\delta(R)}{4\pi DR^2}. \tag{4.4.18}$$

To prove (4.4.18) we must show

$$R^2\nabla^2\frac{1}{R} = 0 \quad \text{if} \quad R \neq 0. \tag{4.4.19}$$

and

$$\int_{\text{small sphere}} \nabla^2\frac{1}{|r-r'|}\, d^3\bar{R} = -4\pi. \tag{4.4.20}$$

The first statement follows from substitution of (4.4.8). For the latter statement we use Gansz' theorem:

$$\int_{\Delta V} \nabla^2\frac{1}{|r-r'|}\, d^3\bar{R} = \int \text{grad}\frac{1}{|r-r'|}\, dS. \tag{4.4.21}$$

Now $\text{grad}(1/|r-r'|) = -1/\bar{R}^2$ and $dS = 4\pi\bar{R}^2$, which proves (4.4.20).

We notice that the solutions

$$G_\varkappa(R) = \frac{e^{\pm i\varkappa R}}{4\pi RD}$$

satisfy the equation

$$(\nabla^2+\varkappa^2)G_\varkappa(R) = -\frac{1}{D}\delta(\mathfrak{r}-\mathfrak{r}').$$

Using the condition of equations(4.4.12a) and (4.4.12b), we find that the condition (4.4.12a) is satisfied by both solutions but using equation (4.4.12b) we find

$$\lim_{\varkappa\to-\infty+i\infty}\frac{e^{\pm i\varkappa R}}{4\pi RD} = \frac{e^{\pm R(-i\infty-\infty)}}{4\pi RD} \tag{4.4.22}$$

and the condition is satisfied only by the solution

$$G_\varkappa(R) = \frac{e^{+i\varkappa R}}{4\pi RD} \tag{4.4.23}$$

which is the required solution for the Green's function.

For the fluctuating variable X, the noise spectrum is given by

$$S_X(\omega) = \frac{4\mathrm{var}X}{V_s} \mathrm{Re} \int_{D_s} \int_{D_s} G(\underset{\sim}{r}, i\omega, \underset{\sim}{r}')\, dV\, dV' \tag{4.4.24}$$

The required double integration sought for the noise spectrum is

$$I = \int_{V_s} \int_{V_s} d^3r\, d^3r'\, G_\varkappa(R). \tag{4.4.25}$$

Further R can be written as

$$R = \sqrt{r^2+r'^2-2rr'\cos\theta} \tag{4.4.26}$$

where θ is the angle between $\underset{\sim}{r}$ and $\underset{\sim}{r}'$. Using polar coordinates with r' as polar axis and integrating (4.4.25) first over r we need the integral (denoting by I_1)

$$I_1 = \iiint r^2 dr \sin\theta\, d\theta\, d\varphi\, \frac{e^{i\varkappa\sqrt{r^2+r'^2-2rr'\cos\theta}}}{4\pi D\sqrt{r^2+r'^2-2rr'\cos\theta}} . \tag{4.4.27}$$

Further we write this integral as

$$I_1 = -\int r^2 dr \int d\varphi \int d(\cos\theta)\, \frac{e^{i\varkappa\sqrt{r^2+r'^2-2rr'\cos\theta}}}{4\pi D\sqrt{r^2+r'^2-2rr'\cos\theta}} . \tag{4.4.28}$$

Realizing that integration over φ contributes 2π and further that

$$\frac{d}{d(\cos\theta)} e^{i\varkappa\sqrt{r^2+r'^2-2rr'\cos\theta}} = \frac{-i\varkappa rr' e^{i\varkappa\sqrt{r^2+r'^2-2rr'\cos\theta}}}{\sqrt{r^2+r'^2-2rr'\cos\theta}} , \tag{4.4.29}$$

we find that (4.4.28) reduces to

$$I_1 = \frac{1}{2\varkappa D\, r' i} \int_0^a r dr\, [e^{i\varkappa(r+r')} - e^{i\varkappa|r'-r|}]. \tag{4.4.30}$$

As to the double integration (4.4.25), we can write

$$\int_0^a \int_0^a f(r,r')dr\, dr' = \int_0^a dr' \int_0^{r'} dr\, f(r,r') + \int_0^a dr' \int_{r'}^a dr\, f(r,r')$$

and it can be shown that the second contribution equals the first one, i.e.

$$\int_0^a \int_0^a dr\, dr'\, f(r,r') = 2 \int_0^a dr' \int_0^{r'} dr\, f(r,r') \tag{4.4.31}$$

because of the symmetry of the integrand $f(r,r')$. We can thus restrict ourselves to $r \leq r'$ and omit the absolute sign in (4.4.30).

Realizing the fact that integration over the solid angle $d\Omega$ contributes 4π, the required double integration yields

$$I = \frac{8\pi}{2\varkappa Di} \int_0^a r'dr' \left[\frac{re^{i\varkappa(r+r')}}{i\varkappa} + \frac{e^{i\varkappa(r+r')}}{\varkappa^2} + \frac{re^{i\varkappa(r'-r)}}{i\varkappa} - \frac{e^{i\varkappa(r-r')}}{\varkappa^2} \right]_0^{r'} \tag{4.4.32}$$

or

$$I = \frac{4\pi}{\varkappa Di} \int_0^a r'dr' \left[\left(\frac{r'}{i\varkappa} + \frac{1}{\varkappa^2}\right) e^{2i\varkappa r'} + \left(\frac{r'}{i\varkappa} - \frac{1}{\varkappa^2}\right) \right]. \tag{4.4.33}$$

Using the following relations due to integration by parts

$$\int r'dr'\, e^{2i\varkappa r'} = \frac{r'e^{2i\varkappa r'}}{2i\varkappa} + \frac{e^{2i\varkappa r'}}{4\varkappa^2} \tag{4.4.34}$$

and

$$\int r'^2 dr' e^{2i\varkappa r'} = \frac{r'^2 e^{2i\varkappa r'}}{2i\varkappa} + \frac{r'e^{2i\varkappa r'}}{2\varkappa^2} - \frac{1}{4i\varkappa^3} e^{2ei\varkappa r'}, \tag{4.4.35}$$

the required integration is finally found to be

$$I = \frac{4\pi}{\varkappa Di}\left[\frac{a^2e^{2i\varkappa a}}{-2\varkappa^2} + \frac{ae^{2i\varkappa a}}{i\varkappa^3} + \frac{e^{2i\varkappa a}}{2\varkappa^4} + \frac{a^3}{3i\varkappa} - \frac{a^2}{2\varkappa^2} - \frac{1}{2\varkappa^4}\right] \quad (4.4.35)$$

Using for the noise spectrum equation (4.4.24) and representing the fluctuating variable as p_0, with $\frac{\Delta P^2}{V_s} = \frac{P_0}{V_s} = p_0$, the number of holes per unit volume, we get

$$S_{\Delta P}(\omega) = \mathrm{Re}\frac{64\pi p_0 a^5}{D\varphi^5}\left[e^{\varphi}(\varphi^2-4\varphi+4)+(\frac{\varphi^3}{3} + \varphi^2-4)\right] \quad (4.4.36)$$

where $\varphi = 2i\varkappa a$.

Equation (4.4.36) can be rewritten as

$$S_{\Delta P}(\omega) = \mathrm{Re}\frac{64\pi p_0 a^5}{D\varphi^5}\,[4e^{\varphi}-4\varphi e^{\varphi}+\varphi^2 e^{\varphi}+\varphi^2+ \frac{\varphi^3}{3} - 4]. \quad (4.4.37)$$

For the required noise spectrum we need to evaluate the real part of expression (4.4.37). We shall discuss two different cases.

4.4.1 Pure Diffusion ($\alpha=0$)

If no sinks or sources are present then $\alpha=0$. In this case the required expression for the noise spectrum, taking the real part of equation (4.4.37), becomes

$$S_{\Delta P}(\omega) = \frac{16\pi a^5 p_0}{D\gamma^5}\{-2+\gamma^2+e^{-\gamma}[\gamma^2(\sin\gamma+\cos\gamma)+4\gamma\cos\gamma+2(\cos\gamma-\sin\gamma)]\}, \quad (4.4.38)$$

where $\gamma = 2a\sqrt{\frac{\omega}{2D}}$.

The result (4.4.38) is the same as previously obtained by van Vliet and Fassett[9] for the case $\alpha=0$. The low and high frequency asymptotes are found to be (expanding sin and cos terms)

$$S_{\Delta P}(\omega) \underset{\omega\to 0}{\to} \frac{32\pi p_0 a^5}{15D} \tag{4.4.39a}$$

and

$$S_{\Delta P}(\omega) \underset{\omega\to\infty}{\to} \frac{p_0 (2D)^{1/2} 4\pi a^2}{\omega^{3/2}} \quad . \tag{4.4.39b}$$

The plot for the noise spectrum is shown in the $\eta=0$ curve of fig.1. Spectrum is constant at low frequencies and at high frequencies the spectrum goes as $\omega^{-3/2}$. The computer program used to plot the spectrum is given in the Appendix. ($u=\omega\tau_d$, $\eta=\alpha\tau_d$).

4.4.2 Diffusion With Sinks ($\alpha\neq 0$)

When sinks are present α is not equal to zero. The required noise spectrum expression, taking real part of (4.4.37) becomes

$$S_{\Delta P}(\omega) = \frac{16\pi a^5 p_0(\beta^2-\delta^2)}{3D(\beta^2+\delta^2)^2} + \frac{16\pi a^5 p_0 K}{D(\beta^2+\delta^2)^5} \tag{4.4.40}$$

where

$$\begin{aligned} K = {} & (10\beta^2\delta^3-5\beta^4\delta-\delta^5)[2\beta\delta+2\delta e^{-\beta}\cos\delta(\beta+2)+e^{-\beta}\sin\delta(\delta^2-\beta^2-4-4\beta)] \\ & -(\beta^5-10\beta^3\delta^2+5\beta\delta^4)[\beta^2-\delta^2+2\delta e^{-\beta}\sin\delta(\beta+2)+e^{-\beta}\cos\delta(\beta^2-\delta^2+4-4\beta)-4] \end{aligned} \tag{4.4.41}$$

$$\beta = \frac{2a}{\sqrt{D}}\left[\sqrt{\tfrac{1}{2}\alpha + \tfrac{1}{2}\sqrt{\alpha^2+\omega^2}}\ \right] \tag{4.4.42}$$

and

$$\delta = \frac{2a}{\sqrt{D}}\left[\sqrt{-\tfrac{1}{2}\alpha + \tfrac{1}{2}\sqrt{\alpha^2+\omega^2}}\ \right]. \tag{4.4.43}$$

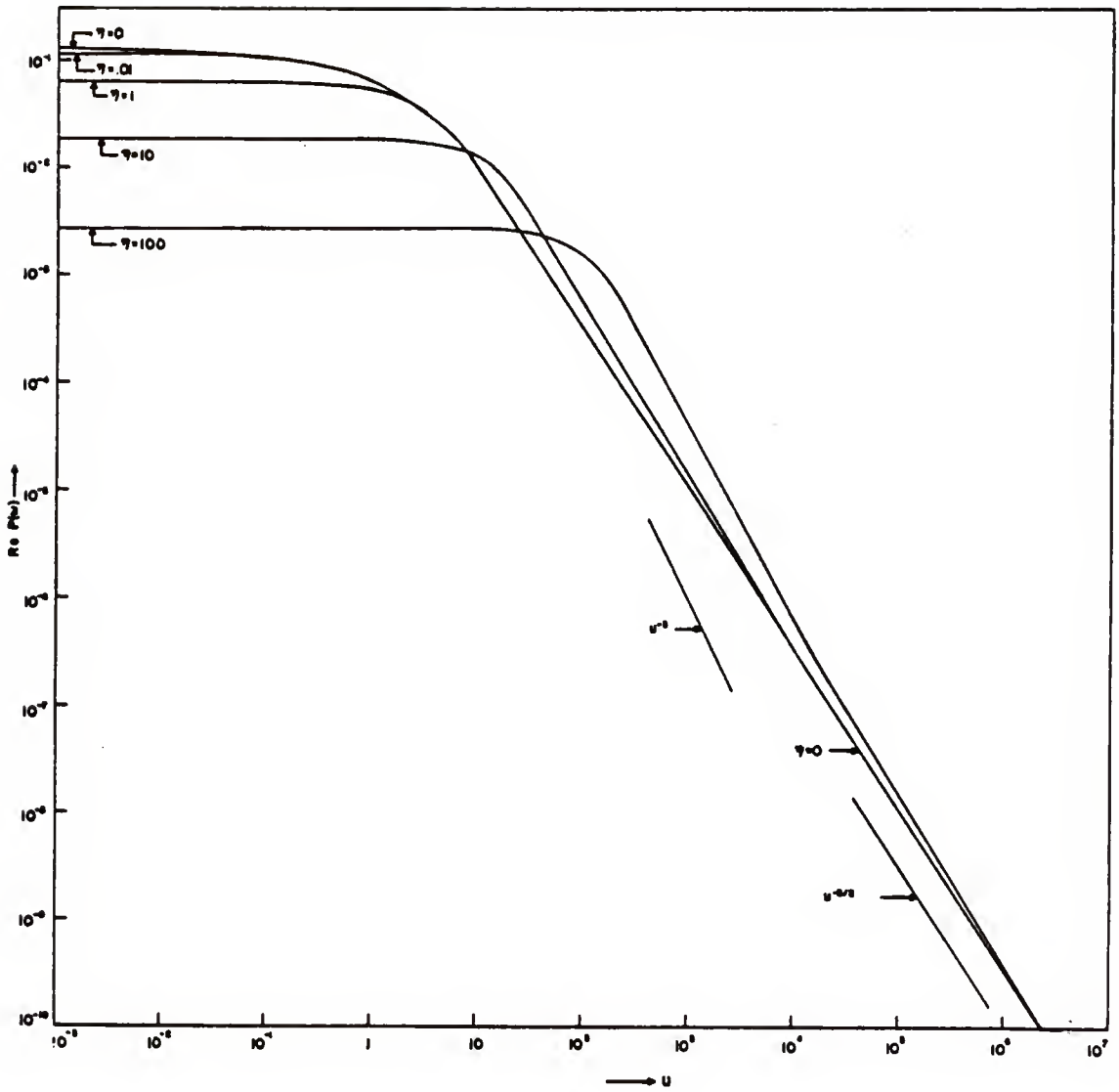


Fig. 1

Noise spectra for embedded sphere ($\eta=0$, $\eta\neq0$)

Furthermore using the following dimensionless quantities

$$\tau_d = a^2/D, \qquad \text{diffusion time}$$

$$u = \omega\tau_d \tag{4.4.44}$$

and

$$\eta = \tau_d/\tau_v = \alpha\tau_d$$

we get for β and δ

$$\beta = \sqrt{2}\sqrt{\eta + \sqrt{\eta^2+u^2}} \tag{4.4.45a}$$

and

$$\delta = \sqrt{2}\sqrt{-\eta + \sqrt{\eta^2+u^2}} \quad . \tag{4.4.45b}$$

If in the above expression $\eta = 0$, we have $\beta = \delta = \gamma$ and equation (4.4.40) reduces to equation (4.4.38). The general case $\alpha \neq 0$ has been considered for the first time as far as we know. The low and high frequency asymptotes are found to be the same as for $\alpha = 0$

$$S_{\Delta P}(\omega) \underset{\omega\to 0}{\to} \frac{32\pi p_0 a^5}{15D} \tag{4.4.46}$$

and

$$S_{\Delta P}(\omega) \underset{\omega\to\infty}{\to} \frac{p_0(2D)^{1/2}4\pi a^2}{\omega^{3/2}} \quad . \tag{4.4.47}$$

The noise spectrum has been plotted in fig.1 . For low frequencies the spectrum is constant and at high frequencies, the spectrum goes as $\omega^{-3/2}$. For $\eta >> 1$ if a range can be found such that $\eta \leq u \leq \eta^3$, then the spectrum goes as ω^{-2} and at higher frequencies $\omega^{-3/2}$ behavior takes over. The computer program used to plot the spectrum is given in the Appendix.

CHAPTER 5
NONSYMMETRICAL EMBEDDED BODIES

5.1 Introduction

We have seen in chapter 2 that for nonsymmetrical geometries Green's function cannot be found in closed form and therefore eigenfunction expansion methods should be used. In this chapter we shall study the noise spectra arising from nonsymmetrical embedded bodies for both Voss-Clarke's[19] 'P' source and the physical diffusion noise[15] source.

5.2 Voss and Clarke's 'P' Source

5.2.1 Nonsymmetrical Bar

We consider the diffusion in an infinite ν-dimensional domain with fictitious boundary conditions i.e. embedded case. In all integrations of chapter 2 d^3r is replaced by $d^\nu r$. The normalized eigenfunctions of (2.2.2) are

$$\varphi_k(\underset{\sim}{r}) = \psi_k(\underset{\sim}{r}) = \frac{1}{V^{1/2}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \tag{5.2.1}$$

with $\underset{\sim}{k} = (k_1,\ldots,k_\nu)$ and

$$k_i = \frac{2\pi n_i}{L_i} , \tag{5.2.2}$$

$$n_i = 0,\pm1,\pm2,\ldots \text{ and } V = \prod_{i=1}^{\nu} L_i ; \tag{5.2.3}$$

for embedded case V goes eventually to ∞. The eigenvalues are

$$\lambda_k = \mu_k = Dk^2, \text{ where } k^2 = k_1^2+k_2^2+\ldots+k_\nu^2 . \tag{5.2.4}$$

Since k-values are dense, we can write

$$\sum_{\underset{\sim}{k}} \sim \int d^\nu k \; Z(\underset{\sim}{k}), \quad \text{with} \quad Z(\underset{\sim}{k}) = \frac{V}{(2\pi)^\nu} \tag{5.2.5}$$

being the density of states for a domain of dimension ν. Since all noise expressions are bilinear in φ or ψ, V cancels out for any sum $\sum_{\underset{\sim}{k}}$ in the limit $V \to \infty$. Then, equation (5.2.5) is exact and the eigenfunction expansions simply represent Fourier integrals. Thus, (2.2.31) results in, with $\int_\infty d^\nu k$ denoting integration over infinite ν dimensional $\underset{\sim}{k}$-space,

$$S_A^{bulk}(\omega) = \frac{1}{(2\pi)^\nu} \int_\infty \int_\infty d^\nu k \; d^\nu k' \; \frac{\left| \int_{V_s} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \, d^\nu r \right|^2}{(Dk^2+i\omega)(Dk'^2-i\omega)}$$

$$\times \iint d^\nu r_1 \; d^\nu r_2 \; S_\xi(\underset{\sim}{r}_1,\underset{\sim}{r}_2) \; e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}_1-\underset{\sim}{k}'\underset{\sim}{r}_2)} \tag{5.2.6}$$

For Voss and Clarke's P source

$$S_\xi(\underset{\sim}{r}_1,\underset{\sim}{r}_2) = P_0^2 \delta(\underset{\sim}{r}_1-\underset{\sim}{r}_2). \tag{5.2.7}$$

So for (5.2.6) we now obtain

$$S_A^{bulk}(\omega) = \frac{P_0^2}{(2\pi)^\nu} \int_\infty d^\nu k \; \frac{\left| \int_{V_s} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \, d^\nu r \right|^2}{D^2k^4+\omega^2} \; . \tag{5.2.8}$$

Now considering V_s to be a rectangular bar of dimensions $-\frac{L_i}{2} \le x_i \le \frac{L_i}{2}$ $i=1,2,\ldots,\nu$, the result for S_A^{bulk} becomes

$$S_A^{bulk}(\omega) = P_0^2 (V_s)^2 \int_\infty \frac{d^\nu k}{D^2k^4+\omega^2} \prod_{i=1}^{\nu} \frac{\sin^2(\frac{1}{2}k_i L_i)}{(\frac{1}{2}k_i L_i)^2} \; . \tag{5.2.9}$$

For a three dimensional rectangular bar $\nu = 3$, and

$$S_A^{bulk}(\omega) = P_0^2 (V_s)^2 \int_{\infty} \frac{d^3k}{D^2k^4+\omega^2} \prod_{i=1}^{3} \frac{\sin^2(\frac{1}{2}k_i L_i)}{(\frac{1}{2}k_i L_i)^2} . \tag{5.2.10}$$

For a three dimensional rectangular bar the spectrum shows breakpoints at $\omega_i = D/L_i^2$, and for the case $L_1 >> L_2 >> L_3$ (nonsymmetrical) the following frequency ranges are claimed by Voss and Clarke[19],

$$\begin{aligned} &\omega << \omega_1, && S(\omega) \propto \omega^{-1/2} \\ &\omega_1 << \omega << \omega_2, && S(\omega) \propto \omega^{-1} \\ &\omega_2 << \omega << \omega_3, && S(\omega) \propto \omega^{-3/2} \\ &\omega >> \omega_3, && S(\omega) \propto \omega^{-2} . \end{aligned} \tag{5.2.11}$$

Voss and Clarke[19] noted in particular the $1/\omega$ range between limits ω_1 and ω_2.

In an effort to find the breakpoints and the spectral frequency ranges the method of numerical integration was used. The complete analysis is given below and the listing of the computer program which uses the trapezoidal rule for numerical integration is given in the Appendix.

Explicitly, for the three dimensional bar we have

$$S_A^{bulk}(\omega) = \text{const.} \int\int\int_{-\infty}^{\infty} \frac{dk_x\, dk_y\, dk_z}{D^2(k_x^2+k_y^2+k_z^2)^2+\omega^2} \tag{5.2.12}$$

$$\times \frac{\sin^2(\frac{1}{2}k_x L_1)}{(\frac{1}{2}k_x L_1)^2} \times \frac{\sin^2(\frac{1}{2}k_y L_2)}{(\frac{1}{2}k_y L_2)^2} \times \frac{\sin^2(\frac{1}{2}k_z L_3)}{(\frac{1}{2}k_z L_3)^2} .$$

Let us call

$$k_x L_1 = \xi$$
$$k_y L_2 = \eta$$
$$k_z L_3 = \zeta \tag{5.2.13}$$

Using the above transformation, we get as integrand

$$\frac{d\xi\, d\eta\, d\zeta}{L_1 \times L_2 \times L_3} \times \frac{1}{\left(\frac{D}{L_1^2}\xi^2 + \frac{D}{L_2^2}\eta^2 + \frac{D}{L_3^2}\zeta^2\right)^2 + \omega^2}$$

$$\times \left(\frac{\sin \frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \times \left(\frac{\sin \frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \times \left(\frac{\sin \frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 . \tag{5.2.14}$$

Furthermore call

$$\omega_1 = D/L_1^2, \quad \omega_2 = D/L_2^2, \quad \omega_3 = D/L_3^2 , \tag{5.2.15}$$

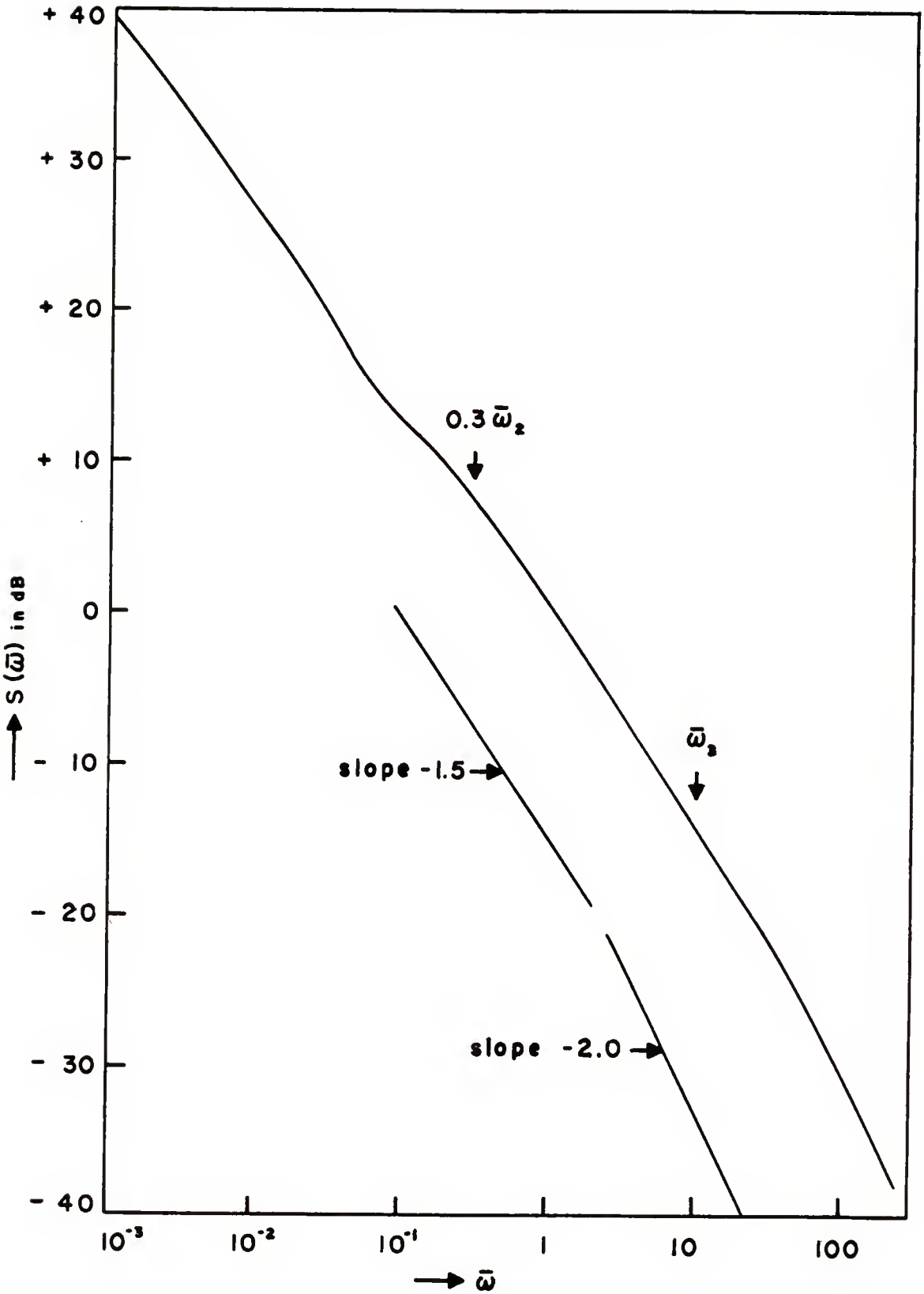
$$\bar{\omega} = \omega/\sqrt[3]{\omega_1\omega_2\omega_3} ,$$

$$\bar{\omega}_1 = \sqrt[3]{\omega_1^2/\omega_2\omega_3} ,$$
$$\bar{\omega}_2 = \sqrt[3]{\omega_2^2/\omega_1\omega_3} , \tag{5.2.16}$$
$$\bar{\omega}_3 = \sqrt[3]{\omega_3^2/\omega_1\omega_2} .$$

We then obtain the integral in the following dimensionless form, with $F(\bar{\omega})$ = const. $S(\bar{\omega})$:

Fig. 2

Noise spectra for nonsymmetrical bar
(Voss and Clarke's 'P' noise source)


+ 40
+ 30
+ 20
+ 10
0
- 10
- 20
- 30
- 40
S ($\bar{\omega}$) in dB
0.3 $\bar{\omega}_2$
$\bar{\omega}_2$
slope -1.5
slope -2.0
10^{-3}
10^{-2}
10^{-1}
1
10
100
$\bar{\omega}$

$$F(\bar{\omega}) = \int\int\int_{-\infty}^{\infty} \frac{d\xi\, d\eta\, d\zeta}{(\bar{\omega}_1\xi^2+\bar{\omega}_2\eta^2+\bar{\omega}_3\zeta^2)^2+\bar{\omega}^2} \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \left(\frac{\sin\frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \left(\frac{\sin\frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 . \tag{5.2.17}$$

Finally we choose as parameters L_1 = 10cm, L_2 = 1cm and L_3 = 0.1cm, D = 25cm^2/sec (corresponding to mobility of 1000 cm^2/volt-sec.) Then we have ω_1 = 0.25rad, ω_2 = 25rad, ω_3 = 2500rad; and $\bar{\omega}_1 = 10^{-2}$, $\bar{\omega}_2 = 1$, and $\bar{\omega}_3 = 10^2$. These are the normalized break-point frequencies. (Units for ω's should read as Rad/sec.)

Substituting all the above constants and using the symmetry property of the integrand we have for (5.2.17)

$$F(\bar{\omega}) = 8\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{d\xi\, d\eta\, d\zeta}{(.01\xi^2+\eta^2+100\zeta^2)^2+\bar{\omega}^2} \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \left(\frac{\sin\frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \left(\frac{\sin\frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 . \tag{5.2.18}$$

The plot of the spectrum is given in fig. 2. We have not been able to verify results as in (5.2.11). The spectrum has no 1/f range and goes much steeper at low frequencies; only the break-points at $\bar{\omega}_2$ and $\bar{\omega}_3$ are visible.

5.3 Physical Diffusion Source

We have seen in chapter 3 that the physical diffusion noise source has a spectrum

$$S_\xi = 4D\, \underset{\sim}{\nabla}\underset{\sim}{\nabla}' n(\underset{\sim}{x})\delta(\underset{\sim}{x}-\underset{\sim}{x}') . \tag{5.3.1}$$

When this is substituted in equation (5.2.6), which is the general result for nonsymmetrical embedded diffusion, one finds for the spectrum

$$S_A^{bulk}(\omega) = C \int_{\infty} \frac{d^{\nu}k \; Dk^2}{D^2k^4+\omega^2} \prod_{i=1}^{\nu} \frac{\sin^2(\frac{1}{2}k_iL_i)}{(\frac{1}{2}k_iL_i)^2} \tag{5.3.2}$$

which differs from equation (5.2.9) by the factor Dk^2 in the numerator; C is a proportionality constant. (For particle diffusion $C = 4n_0/(2\pi)^{\nu}$.)

5.3.1 Rectangular Bar

For a three dimensional rectangular bar equation (5.3.2) becomes

$$S_A^{bulk}(\omega) = C \int_{\infty} \frac{d^3k \; Dk^2}{D^2k^4+\omega^2} \prod_{i=1}^{3} \frac{\sin^2(\frac{1}{2}k_iL_i)}{(\frac{1}{2}k_iL_i)^2} \quad . \tag{5.3.3}$$

Again for the case $L_1 \gg L_2 \gg L_3$ (nonsymmetrical) Voss and Clarke claimed the spectral ranges as

$$\begin{aligned} &f \ll f_1, && S(\omega) = \text{constant} \\ &f_1 \ll f \ll f_2, && S(\omega) \propto -\log f \\ &f_2 \ll f \ll f3, && S(\omega) \propto f^{-1/2} \\ &f \gg f_3, && S(\omega) \propto f^{-3/2} \end{aligned} \tag{5.3.4}$$

Using the same transformation as in section 5.2, the additional factor Dk^2 is found to be

$$\begin{aligned} &= D(k_x^2+k_y^2+k_z^2) \\ &= \frac{D}{L_1^2}\xi^2 + \frac{D}{L_2^2}\eta^2 + \frac{D}{L_3^2}\zeta^2 \\ &= (\omega_1\omega_2\omega_3)^{1/3}(\bar{\omega}_1\xi^2+\bar{\omega}_2\eta^2+\bar{\omega}_3\zeta^2) \quad . \end{aligned} \tag{5.3.5}$$

The overall noise spectrum is given as

$$F(\bar{\omega}) = C \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\xi d\eta d\zeta (\bar{\omega}_1 \xi^2 + \bar{\omega}_2 \eta^2 + \bar{\omega}_3 \zeta^2)}{(\bar{\omega}_1 \xi^2 + \bar{\omega}_2 \eta^2 + \bar{\omega}_3 \zeta^2)^2 + \bar{\omega}^2}$$

$$\times \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \times \left(\frac{\sin\frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \times \left(\frac{\sin\frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 \quad . \tag{5.3.6}$$

For the values of $L_1 = 10\text{cm}$, $L_2 = 1\text{cm}$, $L_3 = 0.1\text{cm}$, and $D = 25\text{cm}^2/\text{sec}$, this becomes

$$F(\bar{\omega}) = C \int_\infty \int \int \frac{d\xi d\eta d\zeta \; (.01\xi^2 + \eta^2 + 100\zeta^2)}{(.01\xi^2 + \eta^2 + 100\zeta^2)^2 + \bar{\omega}^2}$$

$$\times \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \times \left(\frac{\sin\frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \times \left(\frac{\sin\frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 \quad . \tag{5.3.6a}$$

The listing of the computer program is given in the Appendix. We have not been able to verify the results as claimed by Voss and Clarke's equation (5.3.4). The spectrum is given in fig. 3. The high frequency slope is -3/2 as predicted by Lax and Mengert[3]. Breakpoints $\bar{\omega}_2$ and $\bar{\omega}_3$ are visible and at low frequency (not shown) the spectrum levels off. Also there is no obvious 1/f range in the spectrum.

Furthermore we can also see that in the noise spectrum for the nonsymmetrical bar for both the noise sources (namely 'P' source and diffusion source), figs.2 and 3, at the low frequencies there is a "wiggle" in the spectrum. The presence of such a "wiggle" must be attributed to the nonsymmetry of the volume under consideration.

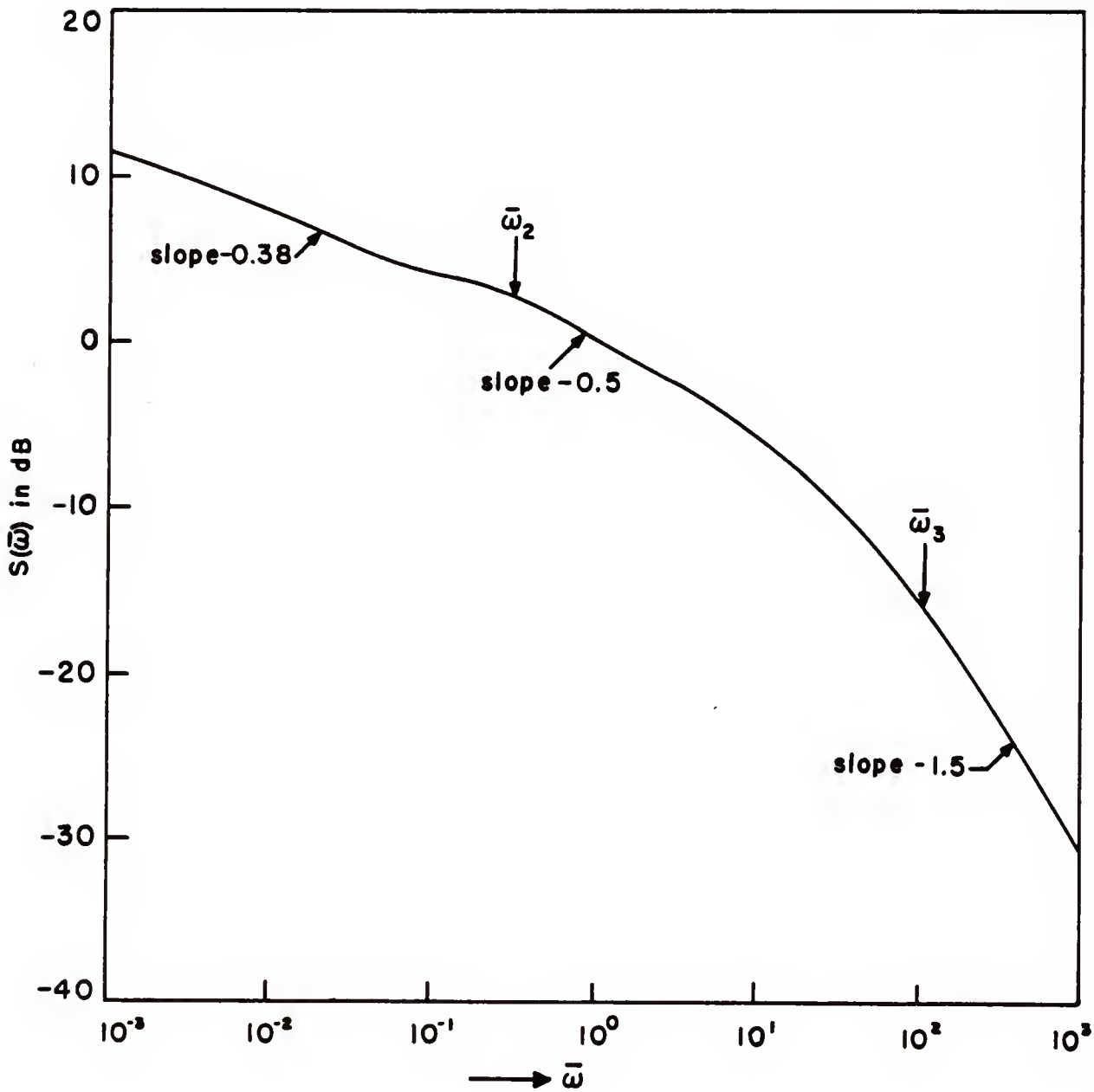


Fig. 3

Noise spectra for nonsymmetrical bar
(Physical diffusion noise source)

The "wiggle" appears on the noise spectrum towards the longest dimension of the specimen. As we shall see in the next subsection, for more symmetrical geometries like square bar and cube, the noise spectrum is smooth and the low frequency "wiggle" disappears (figs. 4 and 5).

5.3.2 Square Disc

The noise spectrum arising from a square disc has been analyzed. In equation (5.3.6) the parameters have been changed. The longest dimension has been eliminated and a square disc of $1cm^2$ cross section and thickness of 0.032cm has been considered. The noise spectrum is given as

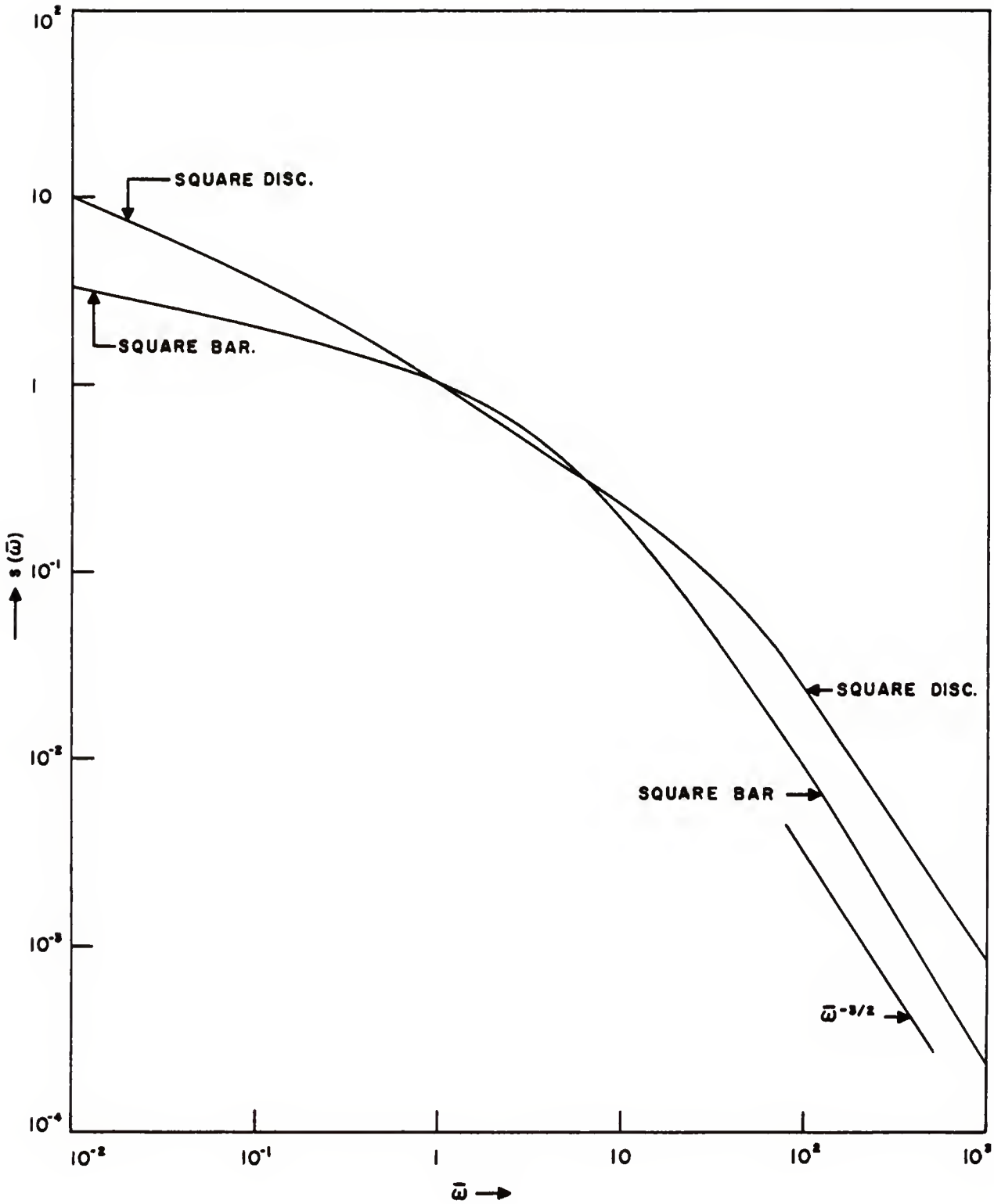
$$F(\bar{\omega}) = C \int_0^\infty \int \int \frac{d\xi d\eta d\zeta (0.1\xi^2+0.1\eta^2+100\zeta^2)}{(0.1\xi^2+0.1\eta^2+100\zeta^2)+\bar{\omega}^2}$$

$$\times \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \times \left(\frac{\sin\frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \times \left(\frac{\sin\frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 \quad . \tag{5.3.7a}$$

The computer program listing is given in the Appendix. The plot for the spectrum is shown in fig. 4. We can see that the spectrum varies smoothly and the low frequency "wiggle" disappears. Further at high frequency the spectrum goes as $f^{-3/2}$. Breakpoints at $\bar{\omega}_1=\bar{\omega}_2$ and $\bar{\omega}_3$ are visible and at low frequencies (not shown) the spectrum levels off.

Fig. 4

Noise spectra for square disc and square bar
(Physical diffusion noise source)


SQUARE DISC.
SQUARE BAR.
SQUARE DISC.
SQUARE BAR
$\bar{\omega}^{-3/2}$
$s(\bar{\omega})$
$\bar{\omega}$
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
1
10
10^2
10^3

5.3.3 Square Rectangular Bar

The noise spectrum resulting from a square rectangular embedded bar has been studied. The smallest dimension has been eliminated and a square bar of crosssection 0.1cm^2 and length 3.16cm has been considered. Equation (5.3.6) for the noise spectrum is

$$F(\bar{\omega}) = C \int_0^\infty \int \int \frac{d\xi d\eta d\zeta (0.01\xi^2+10\eta^2+10\zeta^2)}{(0.01\xi^2+10\eta^2+10\zeta^2)+\bar{\omega}^2}$$

$$\times \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \left(\frac{\sin\frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \left(\frac{\sin\frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 . \qquad (5.3.7)$$

The computer program listing is given in the Appendix. The plot for the spectrum is shown in fig.4 . We see that the spectrum varies smoothly and again the low frequency "wiggle" disappears. At high frequencies the spectrum goes as $\bar{\omega}^{3/2}$. Breakpoints at $\bar{\omega}_1=\bar{\omega}_2$ and $\bar{\omega}_3$ are visible and at low frequencies (not shown) the spectrum levels off.

5.3.4 Cube

The noise spectra obtainable from a less asymmetrical embedded geometry like a cube has been studied. The longest and the smallest dimensions of the nonsymmetrical bar have been eliminated and we considered a cube of dimension 1cm^3. The noise spectra is given by

$$S_A^{bulk}(\bar{\omega}) = C \int\int\int_\infty \frac{d\xi d\eta d\zeta (\xi^2+\eta^2+\zeta^2)}{(\xi^2+\eta^2+\zeta^2)^2+\bar{\omega}^2} \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 \left(\frac{\sin\frac{1}{2}\eta}{\frac{1}{2}\eta}\right)^2 \left(\frac{\sin\frac{1}{2}\zeta}{\frac{1}{2}\zeta}\right)^2 . \qquad (5.3.8)$$

The plot for the noise spectrum is shown in fig. 5. We can see that the spectrum is very smooth, the spectrum levels off at low frequencies and the high frequency slope is $f^{-1.6}$. The break-point at $\bar{\omega}_2$ is quite visible and the roll off is much steeper than for the square bar. The computer program listing is given in the Appendix.

Thus in this chapter we have analyzed the noise spectra obtainable from nonsymmetrical embedded bodies for 'P' source and nonsymmetrical as well as less asymmetrical embedded bodies for the diffusion noise source. We note that the spectral ranges claimed by Voss and Clarke are not verified. For the 'P' source their claim for a 1/f noise spectrum is far from true. Further we have seen that the noise spectra for the physical diffusion noise source results in $f^{-3/2}$ type of behavior at high frequencies. There may be a quasi frequency range where 1/f kind of behavior is observed, but this range is small.

In the next chapters we shall study the effect of the boundary conditions on the noise spectra, both for the symmetrical and non-symmetrical unembedded geometries.

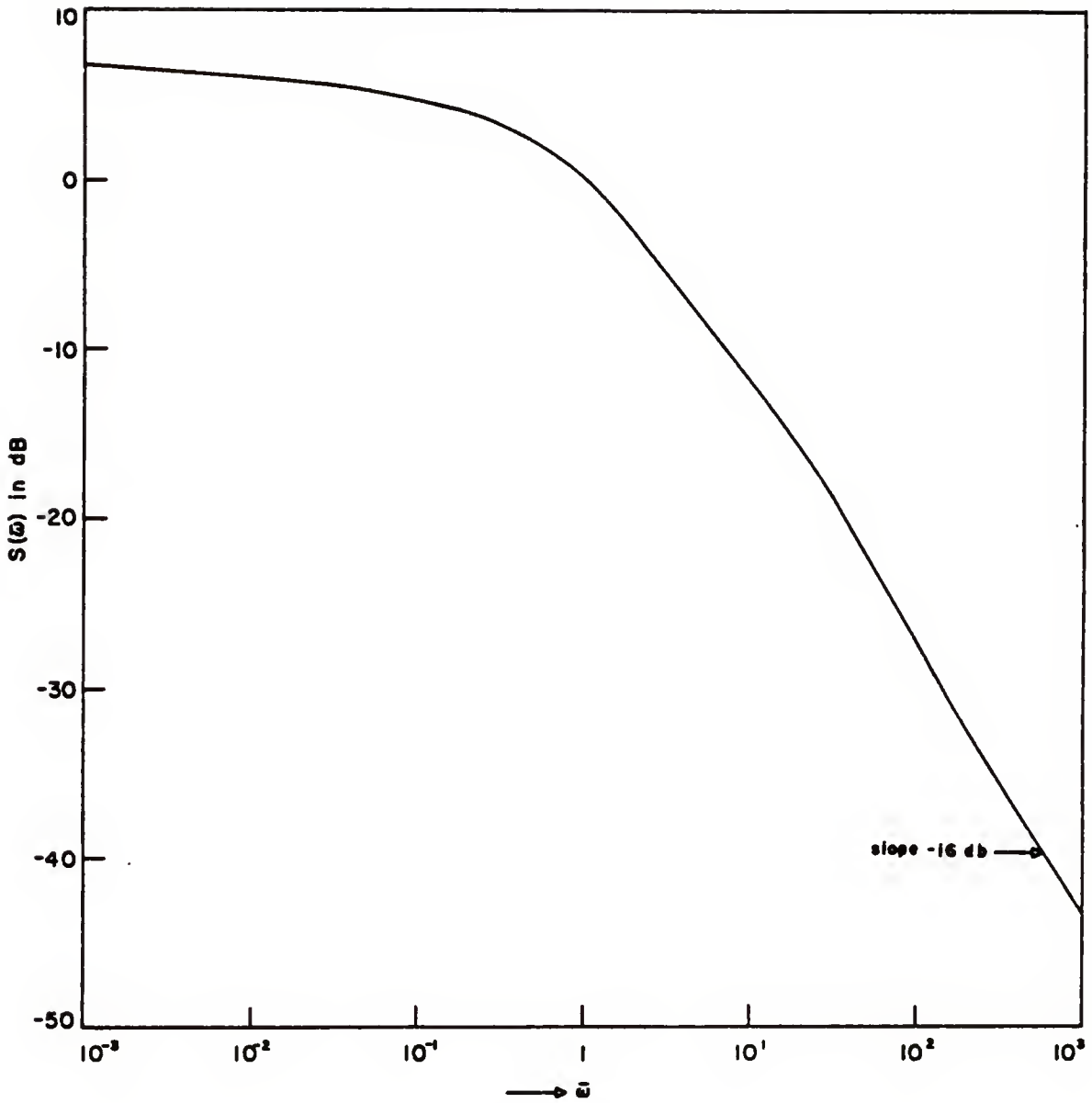


Fig. 5

(Slope indicated is per decade of frequency)

Noise spectra for cube
(Physical diffusion noise source)

CHAPTER 6
SYMMETRICAL NONEMBEDDED BODIES

6.1 Introduction

In this chapter we shall study the noise spectra resulting from symmetrical nonembedded bodies. We already have discussed in chapter 1 the difference between embedded and nonembedded cases, the latter being subject to boundary conditions at the surface of the body. We thus investigate the effects of the boundary conditions on the noise spectra for symmetrical geometries. For example, in the case of surface generation-recombination, we have a mixed Dirichlet-Neumann boundary condition for the fluctuating carrier density Δn,

$$a\Delta n + b\nabla(\Delta n) = 0 \tag{6.1.1}$$

where $a = s$, surface recombination velocity and $b = D$, diffusion constant. The presence of boundary conditions results in (i) altering the shape of the spectra from that of the "infinite medium" and (ii) correlation effects may occur.

6.2 Summary of Previous Work

In this section we briefly review the results of the work done so far to obtain noise spectra from symmetrical nonembedded bodies such as linear bar and cylinder.

6.2.1 Linear Bar

The problem of one dimensional diffusion in a finite domain was studied by van Vliet and Fassett[9]. The one dimensional diffusion in the region $-L \le x \le L$ with perfectly reflecting barriers at $-L$ and $+L$ is considered, where $L \ge d$. The spectral density for the fluctuating variable ΔP, $S_{\Delta P}(\omega)$ in the subregion $-d \le x \le d$ is found to be

$$S_{\Delta P}(\omega) = \frac{p_0(2L)^3}{D} \frac{\sin\theta+\sinh\theta-F(a\theta,b\theta)}{\theta^3(\cosh\theta-\cos\theta)} \tag{6.2.1}$$

where p_0 is the average number of particles per unit length,

$$F(a\theta,b\theta) = \sin a\theta \cosh b\theta + \cos a\theta \sinh b\theta + \sin b\theta \cosh a\theta + \cos b\theta \sinh a\theta$$

and

$$\theta = 2L\sqrt{\omega}/2D, \qquad a = \frac{d}{L}, \qquad b = \frac{L-d}{L} \quad .$$

Equation (6.2.1) reveals that at sufficiently high frequencies the spectrum varies as $\omega^{-3/2}$, whereas at low frequencies the spectrum becomes constant

$$\bar{S}_{\Delta P}(\omega) \underset{\omega\to\infty}{=} \frac{(2D)^{3/2}}{(2L)^3\omega^{3/2}} \tag{6.2.2a}$$

and

$$\bar{S}_{\Delta P}(\omega) \underset{\omega\to 0}{=} \frac{d^2(L-d)^2}{3L^4} \quad . \tag{6.2.2b}$$

Thus the presence of reflecting boundaries limits the low frequency noise, whereas without walls, as in the embedded case the spectrum rises indefinitely as $\omega \to 0$ according to $\omega^{-1/2}$.

6.2.2 Cylinder

The case of fluctuations in the finite cylindrical domain with mixed Dirichlet-Neumann conditions was investigated by van Vliet-Chenette[23] in 1965. Following Morse and Feshbach[24], it was shown that the Green's function or consequently the total spectra can be written as

$$S^{total} = S^{\infty} + S^{\Delta} \tag{6.2.3}$$

where S^{∞}, is the spectrum as in the embedded case and S^{Δ} is due to the presence of boundary conditions. Following this procedure the total spectra for the nonembedded cylinder was found to be

$$S_x^{total} = S_x^{\infty} + S^{\Delta} = 4(\text{var } X)\frac{a^2}{D} \cdot \text{Re } P(\omega) \tag{6.2.4}$$

where

$$P(\omega) = -\frac{1}{(\varkappa a)^2}\{1 + \frac{2\sigma}{\varkappa a}\frac{J_1(\varkappa a)}{D\varkappa J_1(\varkappa a) - \sigma J_0(\varkappa a)}$$

J's are the Bessel functions

$\varkappa = i\sqrt{\frac{i\omega+\alpha}{D}}$, a is the radius

and σ is a constant; in the case of surface recombination, σ is the surface recombination velocity.

The spectrum is constant at low frequencies and at high frequencies the spectrum goes as ω^{-2}, with the presence of a diffusion range $(\omega^{-3/2})$ depending on related time constants. Thus due to the presence of boundary conditions the logarithmic low frequency behavior, as in the embedded case disappears and the spectrum becomes flat.

6.3 Present Work: Sphere

In this section we shall consider the case that the domain $\mathcal{D}_s = \mathcal{D}$ is a spherical domain of radius 'a' with homogeneous boundary conditions on the boundary of the domain

$$D\frac{\partial}{\partial n} G(\underset{\sim}{r}^a, s, \underset{\sim}{r}) + \sigma G(\underset{\sim}{r}^a, s, \underset{\sim}{r}) = 0 \tag{6.3.1}$$

where $\underset{\sim}{r}^a$ is a surface position coordinate, n is the outer normal and σ is the surface recombination velocity. The differential equation for the Green's function is

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + \varkappa^2 G + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]G = -\frac{1}{D}\delta(\underset{\sim}{r}-\underset{\sim}{r}'), \tag{6.3.2}$$

where $G_\varkappa(\underset{\sim}{r},\underset{\sim}{r}') \equiv G(\underset{\sim}{r}, i\omega, \underset{\sim}{r}')$, and $\varkappa$ is the same as in equation (6.2.4) The polar axis we take along $\underset{\sim}{r}'$.

Following the treatment as in *Methods of Theoretical Physics*, vol.1, by Morse and Feshbach[24], page 825, we expand the Green's function in terms of a complete set of functions involving all but one of the coordinates (θ and φ) with coefficients which are undetermined functions of the uninvolved coordinate. Thus let

$$G_\varkappa(\underset{\sim}{r},\underset{\sim}{r}') = \sum_{\ell,m} p_\varkappa^{\ell,m}(r,r')P_\ell^m(\cos\theta)e^{im(\varphi-\varphi')} \tag{6.3.3}$$

$$\ell = 0,1,\ldots,\infty \quad ; \quad m = -\ell,\ldots,+\ell \ .$$

Substituting equation (6.3.3) in (6.3.2) for the first three terms on the left hand side, we get

$$\sum_{\ell,m}\left[\frac{d^2p_\varkappa^{\ell,m}}{dr^2} + \frac{2}{r}\frac{dp_\varkappa^{\ell,m}}{dr} + \varkappa^2 p_\varkappa^{\ell,m}\right]P_\ell^m(\cos\theta)e^{im(\varphi-\varphi')} \tag{6.3.4}$$

and the remaining terms on the left hand side of (6.3.2) give (*Quantum Mechanics*, Schiff[25], page 75)

$$\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]p_\ell^m(\cos\theta)e^{im(\varphi-\varphi')}$$

$$= \frac{-\ell(\ell+1)}{r^2}\, p_m^\ell(\cos\theta)e^{im(\varphi-\varphi')} \quad . \tag{6.3.5}$$

Now expanding the delta function on the right hand side of equation (6.3.2), we get

$$\delta(\mathfrak{r}-\mathfrak{r}') = \frac{\delta(r-r')}{r^2}\,\frac{\delta(\varphi-\varphi')\delta(\theta-\theta')}{\sin\theta} \quad . \tag{6.3.6}$$

Let

$$\frac{\delta(\theta-\theta')\delta(\varphi-\varphi')}{\sin\theta} = \sum_{m,\ell} N_{\ell m}\, F(\theta',\varphi')P_\ell^m(\cos\theta)e^{im\varphi} \tag{6.3.7}$$

where $(N_{\ell m})^{-2}$ is the normalization constant; according to[25] (*Quantum Mechanics*, Schiff, page 73, equation (14.16))

$$N_{\ell m} = \sqrt{\frac{(2\ell+1)}{4\pi}\,\frac{(\ell-|m|)!}{(\ell+|m|)!}} \quad . \tag{6.3.8}$$

Then by inversion from (6.3.7) we get

$$F(\theta',\varphi') = \int \frac{\delta(\theta-\theta')\delta(\varphi-\varphi')}{\sin\theta}\, N_{\ell m}P_\ell^m(\cos\theta)e^{-im\varphi}d\Omega \tag{6.3.9}$$

where $d\Omega = \sin\theta\, d\theta d\varphi$;thus

$$F(\theta',\varphi') = N_{m\ell}P_\ell^m(\cos\theta')e^{-im\varphi'} \quad .$$

Hence we get

$$\delta(\mathfrak{r}-\mathfrak{r}') = N_{\ell m}^2 \sum_{m,\ell} P_\ell^m(\cos\theta)P_\ell^m(\cos\theta')e^{im(\varphi-\varphi')}\,\frac{\delta(r-r')}{r^2} \tag{6.3.10}$$

Since the polar axis is along r', $\theta' = 0$ and

$$P_\ell^m(\cos\theta') = 1. \tag{6.3.11}$$

The result of the full substitution and equating term by term, yields

$$\frac{d^2 p_\varkappa^{\ell,m}}{dr^2} + \frac{2}{r}\frac{dp_\varkappa^{\ell,m}}{dr} + \varkappa^2[1-\frac{\ell(\ell+1)}{r^2}]p_\varkappa^{\ell,m} = -\frac{\delta(r-r')}{Dr^2N_{\ell m}^2}. \tag{6.3.12}$$

The solutions of the homogeneous equation corresponding to (6.3.12) are given by spherical Bessel functions of first and second kind $j_\ell(\varkappa r)$ and $y_\ell(\varkappa r)$ respectively.

The complete solution of equation (6.3.12) is given as

$$p_\varkappa^{\ell,m} = \frac{-N_{\ell m}^2}{r^2 D[W(w_{\ell I}, w_{\ell II})]} \begin{cases} w_{\ell I}(r)w_{\ell II}(r'), & r \le r' \\ w_{\ell I}(r')w_{\ell II}(r), & r \ge r' \end{cases} \tag{6.3.13}$$

where W is the Wronskian evaluated at $r=r'$ and $w_{\ell I}$ and $w_{\ell II}$ are two independent solutions which satisfy the boundary conditions at $r=0$ and $r=a$, respectively. For $r=0$ the solution must be regular. We thus have:

$$w_{\ell I} = j_\ell(\varkappa r) \tag{6.3.14}$$

and

$$w_{\ell II} = Aj_\ell(\varkappa r) + By_\ell(\varkappa r). \tag{6.3.15}$$

$w_{\ell II}$ satisfies the boundary condition (6.3.1) which is

$$D(\frac{\partial w_{\ell II}}{\partial r})_a = -\sigma w_{\ell II}(a). \tag{6.3.16}$$

Substituting $w_{\ell II}$ in equation (6.3.16) we get

$$DA\left[\frac{\partial j_\ell(\varkappa r)}{\partial r}\right]_{r=a} + DB\left[\frac{\partial y_\ell(\varkappa r)}{\partial r}\right]_{r=a} = -\sigma[Aj_\ell(\varkappa a)+By_\ell(\varkappa a)] \qquad (6.3.17)$$

or

$$DB\,\frac{\partial y_\ell(\varkappa a)}{\partial r} + \sigma By_\ell(\varkappa a) = -DA\,\frac{\partial j_\ell(\varkappa a)}{\partial r} - \sigma Aj_\ell(\varkappa a). \qquad (6.3.18)$$

Substituting the recurrence relations, see ref. 26,

$$\frac{\partial y_\ell(\varkappa a)}{\partial r} = \frac{\ell}{a}\,y_\ell(\varkappa a) - \varkappa y_{\ell+1}(\varkappa a) \qquad (6.3.19)$$

and

$$\frac{\partial j_\ell(\varkappa a)}{\partial r} = \frac{\ell}{a}\,j_\ell(\varkappa a) - \varkappa j_{\ell+1}(\varkappa a) \qquad (6.3.20)$$

in equation (6.3.18) and simplifying, we obtain

$$\frac{B}{A} = -\frac{D[\frac{\ell}{a}\,j_\ell(\varkappa a)-\varkappa j_{\ell+1}(\varkappa a)] + \sigma j_\ell(\varkappa a)}{D[\frac{\ell}{a}\,y_\ell(\varkappa a)-\varkappa y_{\ell+1}(\varkappa a)] + \sigma y_\ell(\varkappa a)} \qquad (6.3.21)$$

A suitable choice for the coefficients A and B is:

$$A = i[(\sigma+\frac{D\ell}{a})y_\ell(\varkappa a) - D\varkappa y_{\ell+1}(\varkappa a)] \qquad (6.3.22)$$

and

$$B = -i[(\sigma+\frac{D\ell}{a})j_\ell(\varkappa a) - D\varkappa j_{\ell+1}(\varkappa a)]. \qquad (6.3.23)$$

Substituting A and B in equation (6.3.15) we obtain for $\omega_{\ell II}$ as

$$\begin{aligned} w_{\ell II} = {} & j_\ell(\varkappa r)[(\sigma+\frac{D\ell}{a})iy_\ell(\varkappa a)-D\varkappa iy_{\ell+1}(\varkappa a)] \\ & -iy_\ell(\varkappa r)[(\sigma+\frac{D\ell}{a})j_\ell(\varkappa a)-D\varkappa j_{\ell+1}(\varkappa a)]\ . \end{aligned} \qquad (6.3.24)$$

Now,

$$iy_\ell(\varkappa a) = h_\ell^{(1)}(\varkappa a) - j_\ell(\varkappa a) \tag{6.3.25}$$

where $h_\ell^{(1)}(\varkappa a)$ is the spherical Hankel function of the first kind. Thus we obtain

$$w_{\ell II} = j_\ell(\varkappa r)[(\sigma+\frac{D\ell}{a})h_\ell^{(1)}(\varkappa a)-D\varkappa h_{\ell+1}^{(1)}(\varkappa a)]-h_\ell^{(1)}(\varkappa r)[(\sigma+\frac{D\ell}{a})j_\ell(\varkappa a)-D\varkappa j_{\ell+1}(\varkappa a)] \tag{6.3.26}$$

which can also be written as

$$w_{\ell II} = h_\ell^{(1)}(\varkappa r) - j_\ell(\varkappa r)Q_\ell(\varkappa a) \tag{6.3.27}$$

where

$$Q_\ell(\varkappa a) = \frac{D\varkappa h_{\ell+1}^{(1)}(\varkappa a)-(\sigma+\frac{D\ell}{a})h_\ell^{(1)}(\varkappa a)}{D\varkappa j_{\ell+1}(\varkappa a)-(\sigma+\frac{D\ell}{a})j_\ell(\varkappa a)} . \tag{6.3.28}$$

The Wronskian as defined by

$$W = \begin{vmatrix} w_{\ell I} & w_{\ell I'} \\ w_{\ell II} & w_{\ell II'} \end{vmatrix}$$

(' denotes first derivative) evaluated at $r = r'$ is found to be

$$W(w_{\ell I}, w_{\ell II}) = \frac{i}{\varkappa r'^2} . \tag{6.3.29}$$

Substituting for $r \leq r'$ in equation (6.3.13) we get

$$p_\varkappa^{\ell m}(r,r') = \frac{\varkappa i N_{\ell m}^2}{D}[j_\ell(\varkappa r)h_\ell^{(1)}(\varkappa r') - j_\ell(\varkappa r)j_\ell(\varkappa r')Q_\ell(\varkappa a)] . \tag{6.3.30}$$

Finally, substituting equation (6.3.30) and equation (6.3.8) in equation (6.3.2) we find for the Green's function, for $r \leq r'$

$$G_\varkappa(\mathfrak{x},\mathfrak{x}') = \frac{\varkappa i}{4\pi D} \sum_{\ell,m} \frac{(2\ell+1)(\ell-|m|)!}{(\ell+|m|)!}[j_\ell(\varkappa r)h_\ell^{(1)}(\varkappa r')-j_\ell(\varkappa r)j_\ell(\varkappa r')Q_\ell(\varkappa a)]$$

$$\times P_\ell^m(\cos\theta)e^{im(\varphi-\varphi')} \quad . \tag{6.3.31}$$

For the noise spectrum we must evaluate the six fold integral

$$\bar{S}(\omega) = 2\mathrm{Re} \underset{\substack{6 \text{ fold} \\ \text{reduced} \\ \text{domain}}}{\int\cdots\int} r^2dr\, d(\cos\theta)d\varphi r'^2dr'd(\cos\theta')d\varphi' G_\varkappa(\mathfrak{x},\mathfrak{x}') \tag{6.3.32}$$

where the integral involves the reduced domain $0 < r < r'$, $0 \le r' \le a$. Integration over φ gives 2π for $m=0$; it gives zero for $m\neq 0$. Next we use the addition theorem, ref.25 which gives the relation

$$\frac{e^{i\varkappa R}}{i\varkappa R} = \sum_{\ell=0}^{\infty} (2\ell+1)j_\ell(\varkappa r)h_\ell^{(1)}(\varkappa r)P_\ell(\cos\theta). \tag{6.3.33}$$

Substituting in equation (6.3.31) and separating the noise spectrum into the two parts, indicated by the [ ] of 6.3.31, we find for the first part

$$\bar{S}_I(\omega) = 4\pi\ \mathrm{Re} \underset{5 \text{ fold}}{\int\cdots\int} r^2dr(d\cos\theta)d\varphi'\ r'^2dr'd(\cos\theta')\ \frac{e^{i\varkappa R}}{4\pi RD} \tag{6.3.34}$$

and

$$S_I(\omega) = \frac{4\mathrm{var}\ X}{V_s}\ \bar{S}_I(\omega).$$

The equation (6.3.34) for $S_I(\omega)$ is exactly the same as for the spectrum of the embedded sphere computed in section 4.3; this spectrum will now be denoted as $S_X^\infty(\omega)$.

The second part of the spectrum has an integrand which is regular for $r = r'$, the solution being the same for $r \le r'$ as $r \ge r'$. Hence, we can at once integrate over the full domain $0 \le r \le a$, $0 \le r' \le a$. Thus,

$$\bar{S}_{II}(\omega) = -\mathrm{Re}\,\frac{2\pi\varkappa i}{4\pi D} \underbrace{\int \cdots \int}_{\substack{5\text{ fold,}\\ \text{full}\\ \text{domain}}} r^2 dr d(\cos\theta) d(\cos\theta') r'^2 dr' d\varphi'$$

$$\times \sum_{\ell=0}^{\infty} (2\ell+1) j_\ell(\varkappa r) j_\ell(\varkappa r') Q_\ell(\varkappa a) P_\ell(\cos\theta). \tag{6.3.35}$$

The integration over $d(\cos\theta')d\varphi'$ gives -4π. Further due to the orthogonal property of Legendre functions, integration over $d(\cos\theta)$ contributes -2 only for $\ell=0$:

$$\int_0^\pi P_\ell(\cos\theta) d(\cos\theta) = -\int_{-1}^{1} P_\ell(z) P_0(z) dz = -2\delta_{\ell 0}.$$

Thus we have

$$\bar{S}_{II}(\omega) = -\mathrm{Re}\frac{4\pi\varkappa i}{D} \iint r^2 dr\, r'^2 dr'\, j_0(\varkappa r) j_0(\varkappa r') Q_0(\varkappa a) \tag{6.3.36}$$

($Q_0(\varkappa a)$, substituting $\ell=0$ in equation (6.3.28). We can separate the integrations over r and r'; thus evaluating the integral

$$\int r^2 dr j_0(\varkappa r) = \int r^2 dr \frac{J_{1/2}(\varkappa r)}{\sqrt{\varkappa r}} \sqrt{\frac{\pi}{2}} . \tag{6.3.37}$$

In equation (6.3.37) we have used the relation which connects regular and spherical Bessel functions J and j, which is[26],

$$j_n(z) = \sqrt{\frac{\pi}{2z}}\, J_{n+1/2}(z) . \tag{6.3.38}$$

We can write equation (6.3.37) as

$$\int r^2 dr j_0(\varkappa r) = \int_0^{\varkappa a} \frac{\varkappa^2 r^2 d(\varkappa r) J_{1/2}(\varkappa r)}{\sqrt{\varkappa r}} \sqrt{\frac{\pi}{2}} \frac{1}{\varkappa^3} \tag{6.3.39}$$

$$= \frac{1}{\varkappa^3} \sqrt{\frac{\pi}{2}} \int_0^{\varkappa a} (\varkappa r)^{3/2} d(\varkappa r) J_{1/2}(\varkappa r) \tag{6.3.40}$$

We can also use the derivative formula for the Bessel function which is

$$\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z) \tag{6.3.41}$$

this gives for $\nu = 3/2$ the required integral as

$$\int z^{3/2} J_{1/2}(z) dz = z^{3/2} J_{3/2}(z) \tag{6.3.42}$$

which finally gives for the integral

$$\int r^2 dr j_0(\varkappa r) = \frac{a^3}{(\varkappa a)^{3/2}} \sqrt{\frac{\pi}{2}} J_{3/2}(\varkappa a) \ . \tag{6.3.43}$$

And similarly we get for

$$\int r'^2 dr' j_0(\varkappa r') = \frac{a^3}{(\varkappa a)^{3/2}} \sqrt{\frac{\pi}{2}} J_{3/2}(\varkappa a). \tag{6.3.44}$$

Substituting equations(6.3.43) and (6.3.44) in equation (6.3.36) we get

$$\bar{S}_{II}(\omega) = -\mathrm{Re} \frac{4\pi i a^3}{D\varkappa^2} \frac{\pi}{2} [J_{3/2}(\varkappa a)]^2 Q_0(\varkappa a) \tag{6.3.45}$$

and for the spectrum we obtain

$$S_{II}(\omega) = \frac{4 \mathrm{var} X}{V_s} \bar{S}_{II}(\omega) \ .$$

Taking $V_s = \frac{4}{3}\pi a^3$, we obtain for the fluctuating variable X,

$$S_{II}(\omega) = -\mathrm{Re}\,\frac{6\pi i \mathrm{var}X}{D\varkappa^2}\,[J_{3/2}(\varkappa a)]^2 Q_0(\varkappa a). \tag{6.3.46}$$

Denoting the spectrum associated with second term by $S_X^{\Delta}(\omega)$, we can write the total spectrum for nonembedded sphere as

$$S^{\text{total}}_X(\omega) = S_X^{\infty}(\omega) + S_X^{\Delta}(\omega). \tag{6.3.47}$$

Now we shall derive another form for $S_X^{\infty}(\omega)$ in terms of Bessel functions more suitable for combining with $S_X^{\Delta}(\omega)$. From equation (6.3.34) we can write

$$S_I = \mathrm{Re}\,\frac{8\mathrm{var}X}{V_s} \int\limits_{\substack{6\text{ fold,}\\ \text{reduced}\\ \text{domain}}} r^2 dr\, r'^2 dr'\, d(\cos\theta) d(\cos\theta')\, d\varphi d\varphi'$$

$$\times \sum_{\ell,m} \frac{i\varkappa}{DN_{\ell m}^2}\, j_\ell(\varkappa r) h_\ell^{(1)}(\varkappa r') P_\ell^m(\cos\theta) e^{im(\varphi-\varphi')} \quad ; \tag{6.3.48}$$

integration over $d\varphi$ gives 2π for m=0 only,

integration over $d(\cos\theta)$ gives -2 for ℓ=0 only and

integration over $d(\cos\theta')d\varphi'$ gives -4π.

Thus

$$S_I = \mathrm{Re}\,\frac{8\mathrm{var}X}{\frac{4}{3}\pi a^3}\,\frac{4\pi i\varkappa}{D} \int\limits_0^a r'^2 dr' h_0^{(1)}(\varkappa r') \int\limits_0^{r'} r^2 dr j_0(\varkappa r). \tag{6.3.49}$$

These integrals are similar as found in equation (6.3.37), thus

$$\int\limits_0^{r'} r^2 dr j_0(\varkappa r) = \frac{r'^{3/2}}{\varkappa^{3/2}} \sqrt{\frac{\pi}{2}}\, J_{3/2}(\varkappa r'). \tag{6.3.50}$$

So we get for S_I

$$S_I = \mathrm{Re}\,\frac{8\mathrm{var}X}{V_s}\,\frac{4\pi i\varkappa}{D}\,\frac{\pi}{2}\int_0^a \frac{(\varkappa r')^2 d(\varkappa r')(\varkappa r')^3}{\varkappa^6(\varkappa r')^{3/2}\sqrt{\varkappa r'}}\,H^{(1)}_{1/2}(\varkappa r')J_{3/2}(\varkappa r') \qquad (6.3.51)$$

$$= \mathrm{Re}\,\frac{8\mathrm{var}X}{V_s}\,\frac{4\pi i}{D}\,\frac{1}{\varkappa^5}\,\frac{\pi}{2}\int_0^a (\varkappa r)^3 d(\varkappa r)H^{(1)}_{1/2}(\varkappa r)J_{3/2}(\varkappa r). \qquad (6.3.52)$$

The required integral was calculated by Luke[27]

$$\int_0^a (\varkappa r)^3 d(\varkappa r)H^{(1)}_{\frac{1}{2}}(\varkappa r)J_{\frac{3}{2}}(\varkappa r) = \frac{(\varkappa a)^4}{6}\{H^{(1)}_{\frac{1}{2}}(\varkappa a)J_{\frac{3}{2}}(\varkappa a) + J_{\frac{5}{2}}(\varkappa a)H^{(1)}_{\frac{3}{2}}(\varkappa a)\}.$$

And using[25] (equation 9.1.27) which is (6.3.53)

$$J_{5/2}(\varkappa a) = -J_{1/2}(\varkappa a) + \frac{3}{\varkappa a}J_{3/2}(\varkappa a), \qquad (6.3.54)$$

and further using the Wronskian relation we found that

$$J_{3/2}(\varkappa a)H^{(1)}_{1/2}(\varkappa a) - J_{1/2}(\varkappa a)H^{(1)}_{3/2}(\varkappa a) = \frac{2i}{\pi\varkappa a}\ . \qquad (6.3.55)$$

Substituting equations (6.3.54) and (6.3.55) we found that for equation (6.3.53) (also using (6.3.38)):

$$= \frac{(\varkappa a)^4}{6}\,\{\frac{2i}{\pi\varkappa a} + \frac{6}{\pi}\,j_1(\varkappa a)h^{(1)}_1(\varkappa a)\ . \qquad (6.3.56)$$

For S_I we obtain finally

$$S_I(\omega) = \mathrm{Re}\frac{A}{\varkappa a}\,[-\frac{1}{3\varkappa a} + i\,j_1(\varkappa a)h^{(1)}_1(\varkappa a)] \qquad (6.3.57)$$

where

$$A \equiv \frac{12(\mathrm{var}X)a^2}{D}\ . \qquad (6.3.58)$$

The spectrum for S_{II} from equation (6.3.46) is

$$S_{II} = -\mathrm{Re}\,\frac{6\pi i \,\mathrm{var}\, X}{D\varkappa^2}\,[J_{3/2}(\varkappa a)]^2 Q_0(\varkappa a) \tag{6.3.59}$$

$$= -\mathrm{Re}\frac{12(\mathrm{var}X)a^2}{D\varkappa a}[\sqrt{\frac{\pi}{2\varkappa a}}\,J_{3/2}(\varkappa a)]^2 Q_0(\varkappa a)$$

$$= -\mathrm{Re}\frac{Ai}{\varkappa a}[j_1(\varkappa a)]^2 Q_0(\varkappa a) \tag{6.3.60}$$

where $Q_0(\varkappa a)$ from equation (6.3.27) for $\ell = 0$

$$Q_0(\varkappa a) = \frac{D\varkappa h_1^{(1)}(\varkappa a) - \sigma h_0^{(1)}(\varkappa a)}{D\varkappa j_1(\varkappa a) - \sigma j_0(\varkappa a)} \quad . \tag{6.3.61}$$

For the total spectrum we now find

$$S_X^{total}(\omega) = \mathrm{Re}\frac{A}{(\varkappa a)}[-\frac{1}{3(\varkappa a)} + i\{j_1(\varkappa a)h_1^{(1)}(\varkappa a) - j_1^2(\varkappa a)\,\frac{D\varkappa h_1^{(1)}(\varkappa a) - \sigma h_0^{(1)}(\varkappa a)}{D\varkappa j_1(\varkappa a) - \sigma j_0(\varkappa a)}\}] \tag{6.3.62}$$

Solving for { } quantity first we get

$$i\{\ \} = ij_1(\varkappa a)\,\frac{-\sigma h_1^{(1)}(\varkappa a)j_0(\varkappa a) + \sigma h_0^{(1)}(\varkappa a)j_1(\varkappa a)}{D\varkappa j_1(\varkappa a) - \sigma j_0(\varkappa a)} \quad . \tag{6.3.63}$$

Using ref. 25 (equation (10.1.31)) we have

$$h_0^{(1)}(\varkappa a)j_1(\varkappa a) - h_1^{(1)}(\varkappa a)j_0(\varkappa a) = \frac{i}{(\varkappa a)^2} \quad . \tag{6.3.64}$$

So we get for { } quantity

$$i\{\ \} = -\frac{1}{(\varkappa a)^2}\,[\frac{\sigma j_1(\varkappa a)}{(D\varkappa j_1(\varkappa a) - \sigma j_0(\varkappa a)}]. \tag{6.3.65}$$

Thus we get for the total spectrum:

$$S_X^{total}(\omega) = -ARe \frac{1}{(\varkappa a)^2} [\frac{1}{3} + \frac{\sigma}{\varkappa a} \frac{j_1(\varkappa a)}{(D\varkappa j_1(\varkappa a)-\sigma j_0(\varkappa a))}]. \tag{6.3.66}$$

Further we know that

$$\varkappa a = ia\sqrt{\frac{i\omega+\alpha}{D}} \tag{6.3.67}$$

$$(\varkappa a)^2 = -a^2(\frac{i\omega+\alpha}{D}). \tag{6.3.68}$$

Introducing the following time constants

$$\tau_v = \frac{1}{\alpha} \quad \text{(volume lifetime)} \tag{6.3.69}$$

$$\tau_d = \frac{a^2}{D} \quad \text{(diffusion time)} \tag{6.3.70}$$

$$\tau_s = \frac{a}{\sigma} \quad \text{(surface lifetime)} , \tag{6.3.71}$$

and also introducing the dimensionless quantities.

$$u = \omega\tau_d, \quad \eta = \tau_d/\tau_v, \quad \xi = \tau_s/\tau_d \tag{6.3.72}$$

we can write

$$(\varkappa a)^2 = -(iu+\eta). \tag{6.3.73}$$

We can rewrite equation (6.3.66) for S_X^{total} as

$$S_X^{total}(\omega) = \overline{A}Re \frac{1}{iu+\eta}[1+ \frac{3\sigma}{D\varkappa^2 a} \frac{j_1(\varkappa a)}{j_1(\varkappa a)-(\sigma/D\varkappa)j_0(\varkappa a)}] , \tag{6.3.74}$$

where $\overline{A} = \frac{A}{3} \equiv \frac{4(\text{var } X)a^2}{D}$. Using the time constants and dimensionless quantities defined in equations (6.3.69) to (6.3.72), we have

$$\frac{\sigma}{D\varkappa} = - \frac{i}{\xi\sqrt{\eta+iu}} , \tag{6.3.75}$$

and

$$\frac{\sigma}{D\varkappa^2 a} = - \frac{1}{\xi(\eta+iu)} . \tag{6.3.76}$$

Using equations (6.3.75) and (6.3.76), the spectrum finally becomes

$$S_X^{total}(\omega) = \frac{4(\text{var } X)a^2}{D} \text{Re}P(\omega) \tag{6.3.77}$$

where

$$P(\omega) = \frac{1}{\eta+iu}\left[1 - \frac{3j_1(i\sqrt{\eta+iu})}{\xi(\eta+iu)j_1(i\sqrt{\eta+iu})+i\sqrt{\eta+iu}\ j_0(i\sqrt{\eta+iu})}\right] . \tag{6.3.78}$$

Now we shall discuss various cases.

6.3.1 No Volume Sinks

If there are no volume sinks present, then $\eta = 0$. For this case

$$\text{Re}P(\omega) = \text{Re}\ \frac{1}{iu}\left[1 - \frac{3j_1(i\sqrt{iu})}{\xi(iu)j_1(i\sqrt{iu})+i\sqrt{iu}\ j_0(i\sqrt{iu})}\right] . \tag{6.3.79}$$

The first term does not contribute. Using the relations for spherical Bessel functions[26] we have

$$j_0(z) = \frac{\sin z}{z} \tag{6.3.80}$$

and

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} . \tag{6.3.81}$$

Thus we obtain

$$\text{Re } P(\omega) = \frac{3(AD+BC)}{4\alpha^4\xi(C^2+D^2)} \tag{6.3.82}$$

where $\alpha = \sqrt{\frac{u}{2}}$

$$A = (-\sin\alpha \cosh\alpha+\alpha \cos\alpha \cosh\alpha+\alpha \sin\alpha \sinh\alpha) \tag{6.3.83}$$

$$B = (\cos\alpha \sinh\alpha + \alpha\sin\alpha \sinh\alpha-\alpha \cos\alpha \cosh\alpha) \tag{6.3.84}$$

$$C = (\cos\alpha \sinh\alpha + \alpha \sin\alpha \sinh\alpha - \alpha \cos\alpha \cosh\alpha - \frac{\cos\alpha \sinh\alpha}{\xi}) \tag{6.3.85}$$

$$D = (-\sin\alpha \cosh\alpha + \alpha \cos\alpha \cosh\alpha + \alpha \sin\alpha \sinh\alpha + \frac{\sin\alpha \cosh\alpha}{\xi}). \tag{6.3.86}$$

At low frequencies, $u \ll 1$, and the spectrum reaches a limiting value. For finite domain a plateau is always reached. Expanding the trigonometric terms, we find canceling terms up to order α^4:

$$\operatorname*{Re}_{\omega\to 0} P(\omega) = \frac{1+5\xi}{15}. \tag{6.3.87}$$

At high frequencies, $u \gg 1$, and the spectrum reaches the asymptote

$$\operatorname*{Re}_{\omega\to\infty} P(\omega) = \frac{3}{\xi\mu^2}. \tag{6.3.88}$$

For a closer investigation we must distinguish between two cases.

6.3.1.(a) . First, let $\xi \ll 1$, or $\tau_s \ll \tau_d$. This option corresponds to the "diffusion limited" case. In equation (6.3.87) the terms in ξ are now irrelevant and the spectrum reaches the low frequency limit $\operatorname{Re} P(0) = \frac{1}{15}$. At high frequencies there is another region, prior to the u^{-2} region, in the range $1 \ll u < 1/\xi^2$.

6.3.1.(b) . The surface limited case occurs when $\xi \gg 1$, or $\tau_s \gg \tau_d$. In this case the diffusion process is practically instantaneous and the noise is only due to surface sinks. At high frequencies the limiting behavior is $\frac{3}{\xi u^2}$ for any $u \gg 1$. However, this spectrum is also found at small u, since in the range $\frac{1}{\xi} < u \ll 1$, we also found the limiting value of the spectrum to be $\frac{3}{\xi u^2}$. This behavior occurs therefore for all $u > \frac{1}{\xi}$. The turnover occurs

for $u \approx \frac{1}{\xi}$ or $\omega \approx \frac{1}{\tau_s}$. The plateau that then is reached has the value $\frac{\xi}{3}$.

The full spectrum for various ξ has been plotted and is shown in fig.6. We see for $\xi >> 1$, or for the surface limited case, the turnover frequency is $u = 2/\xi$. For $\xi << 1$, or for the diffusion limited case, for $\xi = .01$, the diffusion type behavior is observed, in the range $10 < u < 7\times10^3$. The computer program used to plot the noise spectrum is given in the Appendix.

6.3.2 No Surface Sinks

If $\sigma = 0$, $\frac{\partial G}{\partial n} = 0$. This corresponds to ξ being infinite. The spectrum despite the possible presence of microscopic diffusion exhibits only a regular relaxation spectrum.

$$\text{Re } P(\omega) = \frac{\eta}{\eta^2+u^2} \tag{6.3.89}$$

or the spectrum

$$S(\omega) = \frac{4(\text{var } X)\tau_v}{1+\omega^2\tau_v^2} . \tag{6.3.90}$$

This is the "volume limited case".

6.3.3 Both Surface and Volume Sinks Present

When both surface and volume sinks are present we have that $\eta \neq 0$. We again use the relations for j_1 and j_0 as given in equations (6.3.80) and (6.3.81) and after much algebra the real part is found to be

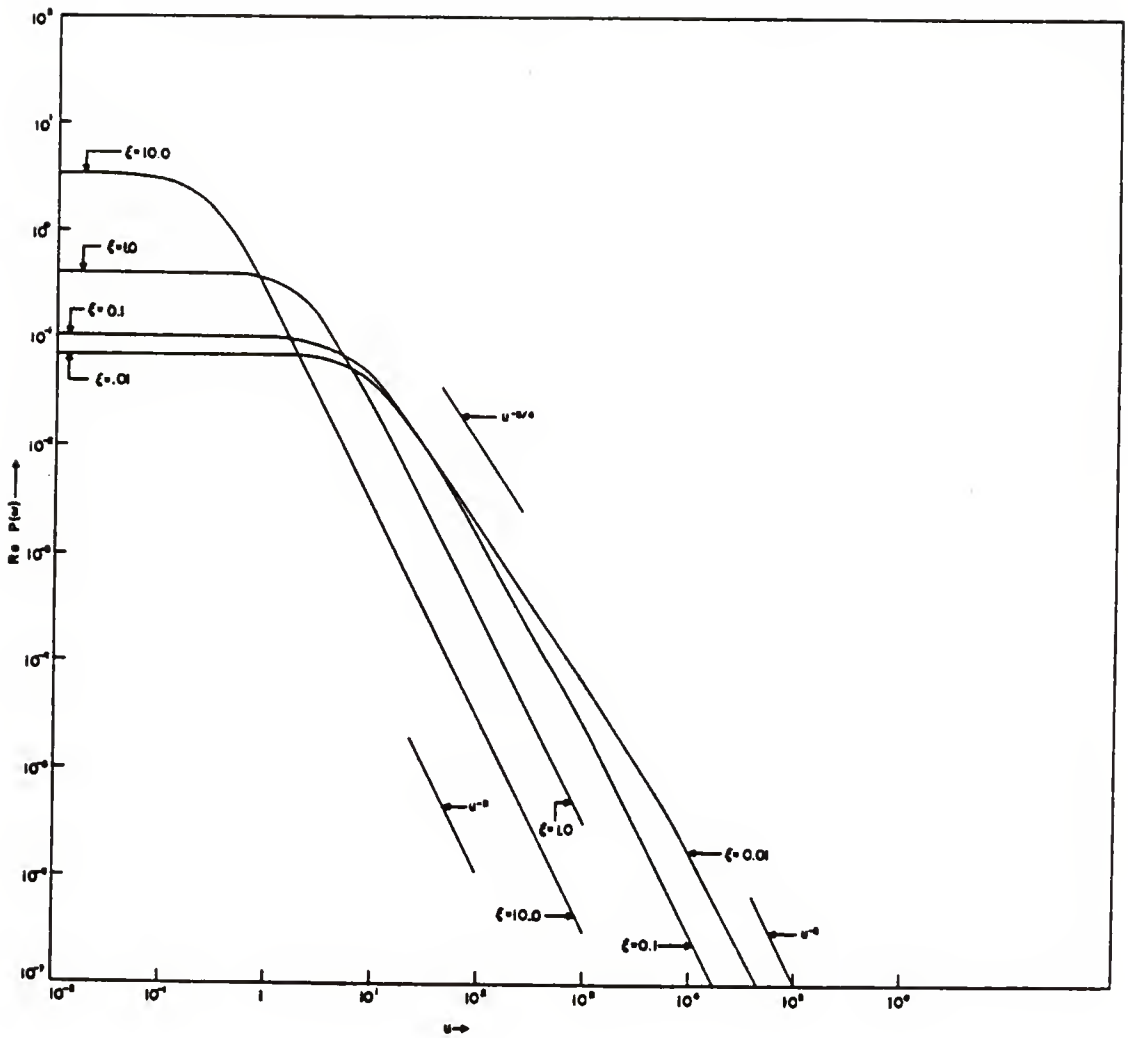


Fig. 6

Noise Spectra for Nonembedded Sphere (η=0)
(Slopes indicated are $u^{-3/2}$ and u^{-2})

$$\mathrm{Re}\ P(\omega) = \frac{\xi(MP+NQ)-3(AM+BN)}{\xi\{(\beta^2P+\delta^2P)^2+(\beta^2Q+\delta^2Q)^2\}} \tag{6.3.91}$$

where

$$\beta = \sqrt{\frac{1}{2}\eta + \frac{1}{2}\sqrt{\eta^2+u^2}} \quad , \tag{6.3.92}$$

$$\delta = \sqrt{-\frac{1}{2}\eta + \frac{1}{2}\sqrt{\eta^2+u^2}} \quad , \tag{6.3.93}$$

$$A = (-\sin\delta\cosh\beta+\delta\cos\delta\cosh\beta+\beta\sin\delta\sinh\beta) \ , \tag{6.3.94}$$

$$B = (\cos\delta\sinh\beta-\beta\cos\delta\cosh\beta+\delta\sin\delta\sinh\beta) \ , \tag{6.3.95}$$

$$C = A + \frac{\sin\delta\cosh\beta}{\xi} \quad , \tag{6.3.96}$$

$$D = B - \frac{\cos\delta\sinh\beta}{\xi} \quad , \tag{6.3.97}$$

$$P = (\beta^2-\delta^2)C-2\beta\delta D \tag{6.3.98}$$

$$Q = (\beta^2-\delta^2)D+2\beta\delta C \tag{6.3.99}$$

$$M = (\beta^2-\delta^2)P - 2\beta\delta Q \tag{6.3.100}$$

$$N = (\beta^2-\delta^2)Q+2\beta\delta P. \tag{6.3.101}$$

At low frequencies in the above equation (6.3.91) $\beta = \sqrt{\eta}$ and $\delta = 0$. The low frequency plateau value for small η is found to be

$$\mathrm{Re}\ \underset{\omega\to 0}{P(\omega)} = \frac{1+5\xi}{15} \tag{6.3.102}$$

as in the previous subsection.

At high frequencies $\beta = \delta = \sqrt{\frac{u}{2}}$. The high frequency limiting value is found to be for any η

$$\mathrm{Re}\ \underset{\omega\to\infty}{P(\omega)} = \frac{3}{\xi u^2} \quad . \tag{6.3.103}$$

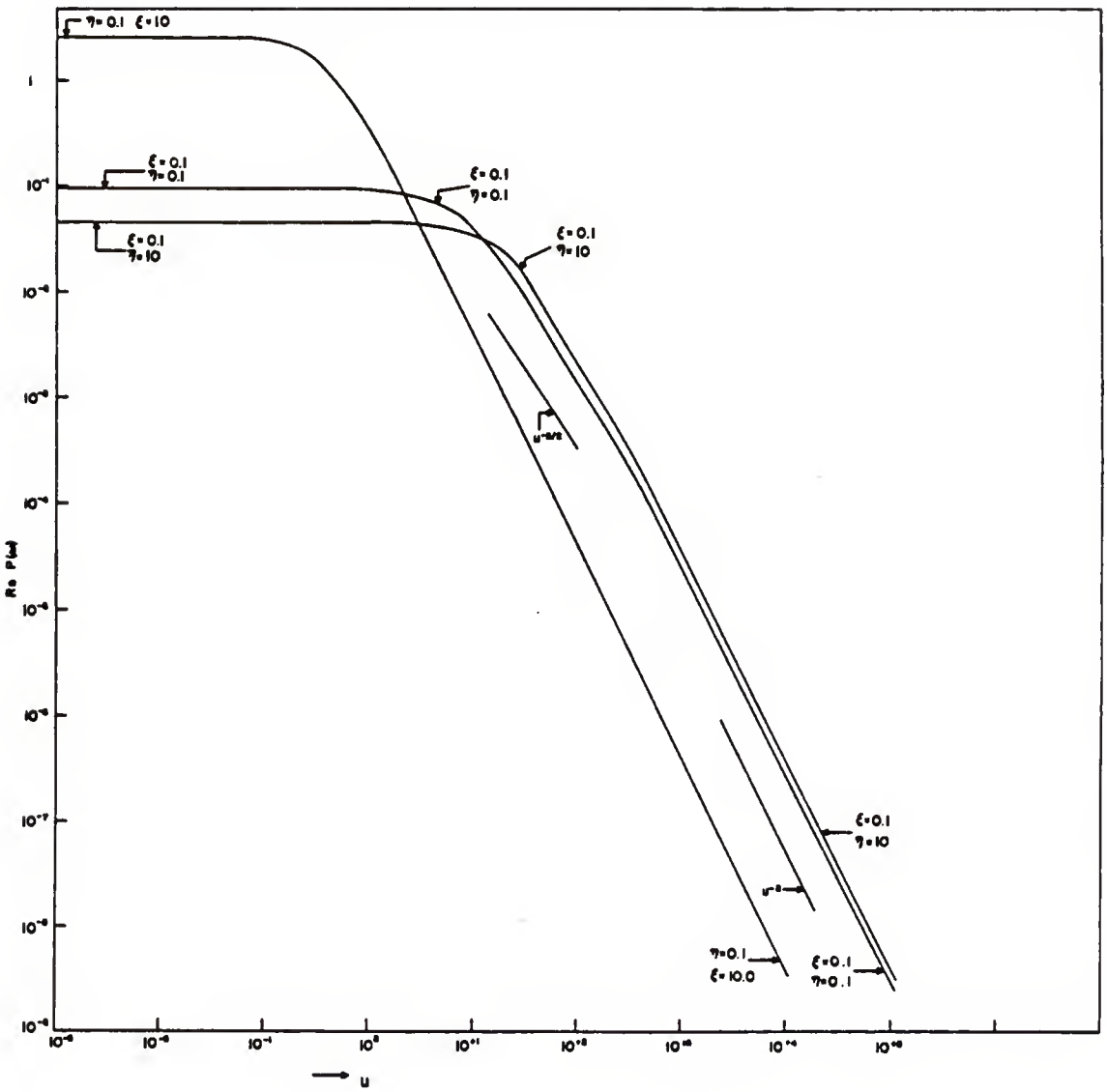


Fig. 7

Noise spectra for nonembedded sphere ($\eta \neq 0$)

For $\eta = 0$ equation (6.3.91) reduces to equation (6.3.82). The noise spectrum is plotted for different values of ξ and η, and is shown in fig.7. We distinguish the following options.

6.3.3(a). For $\xi \gg 1$, or $\tau_s \gg \tau_d$, we have a surface limited spectrum. The low frequency plateau is

$$\mathrm{Re}\ P(0) \approx \frac{3}{\xi} .$$

And, at high frequencies the spectrum goes as u^{-2} and the high frequency limiting value is found to be

$$\mathrm{Re}\ \underset{\omega\to\infty}{P(\omega)} \approx \frac{3}{\xi u^2} ,$$

as predicted.

6.3.3(b). For $\xi \ll 1$ in addition to $\eta \ll 1$ we found a diffusion limited range. The low frequency limit is

$$\mathrm{Re}\ \underset{\omega\to 0}{P(\omega)} = \frac{1+5\xi}{15} .$$

And in the range $1 \ll u < \frac{1}{\xi^3}$ we see that the spectrum goes as $u^{-3/2}$, reflecting the typical diffusion behavior.

For high frequencies the spectra go as u^{-2}. The computer program used to plot the spectrum is given in the Appendix.

CHAPTER 7
NONSYMMETRICAL NONEMBEDDED BODIES

7.1 Introduction

In this chapter we shall analyze the noise spectra resulting from diffusion and generation-recombination in nonsymmetrical nonembedded geometries. We have already seen in Chapter 1 the difference between embedded and nonembedded bodies, the latter being subjected to various boundary conditions. We shall use the boundary conditions stemming from a finite surface recombination velocity at either one pair or two pairs of surfaces of the semiconductor sample under consideration. A three dimensional case is of academic value only, since one always has an ohmic contact in one of the directions.

7.2 Linear Case

The Green's function procedure has been used to find a closed form expression for the noise spectra resulting from samples being subject to a finite surface recombination velocity in the z-direction.

$$D\left(\frac{\partial \Delta p}{\partial z}\right)_{\pm C} = \pm S \Delta p \qquad (7.2.1)$$

where we have assumed that the sample is infinite in x and y direction. D is the diffusion constant, S is the surface recombination velocity and we have considered the sample dimension to be 2C in the z-direction; Δp being the fluctuation in the number of particles, say holes. This problem finds application in noise resulting from buried layers as studied by Hsieh[28]. The diffusion equation is

$$L(\Delta P) = \frac{\partial}{\partial t}(\Delta p) + \frac{\Delta p}{\tau} - \frac{D\partial^2(\Delta p)}{\partial z^2} = 0, \tag{7.2.2}$$

where τ is the carrier lifetime; the Laplace transformed Green's function satisfies the equation

$$G(z,s,z')(s+\frac{1}{\tau}) - D\frac{\partial^2 G(z,s,z')}{\partial z^2} = \delta(z-z'). \tag{7.2.3}$$

According to the standard techniques the Green's may be found by setting

$$G(z,z') = -\frac{1}{DW(u,v)}\begin{cases} u(z)\ u(z') & z \le z' \\ u(z')v(z) & z \ge z' \end{cases} \tag{7.2.4}$$

where W is the Wronskian of two independent solutions u(z) and v(z) which satisfy the left and right side boundary conditions, respectively, evaluated at $z = z'$. The solution is found to be [by Lax and Mengert[3]]

$$G(z,j\omega,z') = (2\gamma D\alpha)^{-1}(\cosh\gamma z+\alpha\sinh\gamma z)(\cosh\gamma z'-\alpha\sinh\gamma z') \tag{7.2.5}$$

for $z \le z'$

and

$$G(z,j\omega,z') = (2\gamma D\alpha)^{-1}(\cosh\gamma z'+\alpha\sinh\gamma z')(\cosh\gamma z-\alpha\sinh\gamma z) \tag{7.2.6}$$

for $z \ge z'$

where

$$\gamma = \sqrt{(1+j\omega\tau)/D\tau} \tag{7.2.6}$$

and

$$\alpha = \frac{\sin\gamma C + (\frac{S}{\gamma D})\cosh\gamma C}{\cosh\gamma C + (\frac{S}{\gamma D})\sinh\gamma C}. \tag{7.2.7}$$

Introducing the surface lifetime and diffusion time by

$$\tau_s = C/S \quad \text{and} \quad \tau_d = C^2/D \tag{7.2.8}$$

and carrying out the required double integration the spectrum is then found to be (see also [9]),

$$S_{\Delta P} = \frac{4N_0P_0}{N_0+P_0} \mathrm{Re}\{\frac{\tau}{1+j\omega\tau} [1- \frac{1}{(\frac{\tau_s}{\tau})(1+j\omega\tau)+\gamma C \coth\gamma C}]\} \tag{7.2.9}$$

where

$$\gamma C = [\frac{\tau_d}{\tau(1+j\omega\tau)}]^{1/2} . \tag{7.2.10}$$

A critical discussion of this result is found both in Champlin's[2] paper and in the Lax-Mengert paper. The spectrum is constant at low frequencies. If a range exists such that $8\tau_d/\tau^2 < \omega < \tau_d/2\tau_s^2$, the spectrum goes as $\bar{\omega}^{3/2}$, whereas at high frequencies the spectrum goes as $\bar{\omega}^2$ and is dominated by the surface lifetime τ_s.

7.3 Rectangular Nonembedded Bar

This kind of problem has been attacked by Lax-Mengert[3], van Vliet-van der Ziel[29] and Champlin[2]. The agreement between various methods of solutions has been shown by van Vliet[4].

The equation governing the flow of minority carriers (holes) is the ambipolar transport equation

$$\frac{\partial}{\partial t} (\Delta p) = - \frac{\Delta p}{\tau} - \mu E\cdot\nabla(\Delta p) + D\nabla^2(\Delta p) \tag{7.3.1}$$

with the boundary conditions in y and z directions

$$D(\frac{\partial \Delta p}{\partial y})_{\pm B} = \pm S\Delta p$$

and (7.3.2)

$$D(\frac{\partial \Delta p}{\partial z})_{\pm C} = \pm S\Delta p ,$$

where we have assumed that the sample is infinite in the x-direction and that the noise is measured between probes a distance 2A apart. D is the diffusion constant, S is the surface recombination velocity and we have considered the sample dimension 2B and 2C in y and z directions respectively. Further if no sweep effects occur, we may assume that the concentration disturbances to be independent of x. Since there is no field in y and z directions equation (7.3.1) becomes

$$L(\Delta P) \equiv \frac{\partial(\Delta P)}{\partial t} + \frac{\Delta P}{\tau} - \frac{D\partial^2(\Delta P)}{\partial y^2} - \frac{D\partial^2(\Delta P)}{\partial z^2} = 0. \tag{7.3.3}$$

We substitute

$$\hat{L} = S - \frac{1}{\tau} - \Lambda$$

and (7.3.4)

$$\Lambda(y\ z) = D\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) .$$

Note that Λ is slightly differently defined as in Chapter 2. Thus we must solve for the eigenfunctions of Λ , i.e. of the two dimensional diffusion equation

$$D\left(\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) = \lambda\varphi \quad . \tag{7.3.5}$$

Subject to boundary conditions as given in (7.3.2), the normalized eigenfunctions are found to be

$$\varphi_{ij} = \frac{4\sqrt{\xi\eta}\ \cos\left(\frac{\xi_i y}{B}\right)\cos\left(\frac{\eta_j z}{C}\right)}{V_s(2\xi_i+\sin 2\xi_i)^{1/2}(\eta_j+\sin 2\eta_j)^{1/2}} \tag{7.3.6}$$

where V_s is the volume of the sample 8ABC. The eigenvalues are given by

$$\lambda_{ij} + D\xi_i^2/B^2 + D\eta_j^2/C^2 = 0 \; ; \tag{7.3.7}$$

here ξ_i and η_j are the solutions of the transcendental equations

$$\xi \tan\xi = SB/D$$

and (7.3.8)

$$\eta \tan\eta = SC/D$$

So the Green's function is given by (see equation (2.2.41))

$$G(y,z,s,y',z') = \sum_{i,j} \frac{\varphi_{ij}(y,z)\,\varphi_{ij}(y',z')}{-\lambda_{ij}+S+\frac{1}{\tau}} \tag{7.3.9}$$

and the spectrum of the total hole fluctuation ΔP is obtained by the standard procedure (by van Vliet and Fassett[9])

$$S_{\Delta P}(\omega) = \frac{4N_0P_0}{N_0+P_0} \sum_{i,j} \frac{|F_{ij}|^2\left(\frac{1}{\tau} + \frac{D\xi_i^2}{B^2} + \frac{D\eta_j^2}{C^2}\right)}{\omega^2+\left(\frac{1}{\tau} + \frac{D\xi_i^2}{B^2} + \frac{D\eta_j^2}{C^2}\right)^2} \; . \tag{7.3.10}$$

The normalized spectrum is found by letting $\omega\tau = \bar{\omega}$ and multiplying equation (7.3.10) by τ, writing $\bar{\omega}_y = D\tau/B^2$, $\bar{\omega}_z = D\tau/C^2$

$$S_{\Delta P}(\bar{\omega}) = \text{const} \sum_{i,j} \frac{|F_{ij}|^2(1+\bar{\omega}_y\xi_j^2 + \bar{\omega}_z\eta_j^2)}{\bar{\omega}^2+(1+\bar{\omega}_y\xi_i^2 + \bar{\omega}_z\eta_j^2)^2} \tag{7.3.11}$$

with

$$F_{ij} = \int_{-B}^{B}\int_{-C}^{C} \frac{\varphi_{ij}(y,z)}{\sqrt{8ABC}}\, dy\, dz$$

$$= \frac{4}{\sqrt{\xi_i\eta_j}} \frac{\sin\xi_i \sin\eta_j}{(2\xi_i+\sin 2\xi_i)^{1/2}(2\eta_j+\sin 2\eta_j)^{1/2}} \; . \tag{7.3.12}$$

A particular simple form is obtained if $S = \infty$, (Dirichlet boundary conditions, Δp (surface) = 0), then,

$$\xi_i = \eta_j = \frac{\pi}{2}(2i+1), \quad i=0,1,\ldots \tag{7.3.13}$$

and

$$S_{\Delta P}(\omega) = \frac{4N_0P_0}{N_0+P_0}\frac{64}{\pi^4}\sum_{i,j=0}^{\infty}\frac{\frac{1}{\tau}+\frac{D\pi^2(2i+1)^2}{4B^2}+\frac{D\pi^2(2j+1)^2}{4C^2}}{(2i+1)(2j+1)\left[\omega^2+\left(\frac{1}{\tau}+\frac{D\pi^2(2i+1)^2}{4B^2}+\frac{D\pi^2(2j+1)^2}{4C^2}\right)^2\right]} \tag{7.3.14}$$

But besides the remarkable feature that the low frequency plateau value is depressed by a factor $64/\pi^4 \approx 0.66$, compared to pure bulk noise, not much can be learned from the result.

The computer program for (7.3.11) has been given in the Appendix, which first solves for the transcendental equation for ξ and η and then solves for the spectrum. The values of B and C were taken to be 0.5 and 0.5 or 0.05 and 0.5 cm respectively; D was taken to be 25 cm^2/sec and τ as 10^{-3} and 10^{-6} sec; for S we took 500 cm/sec. The plot for the noise spectrum is shown in fig. 8.

The noise spectrum is constant at low frequencies and smoothly turns over to f^{-2} kind of behavior at high frequencies, though the turnover is slower than for a bulk g-r spectrum.

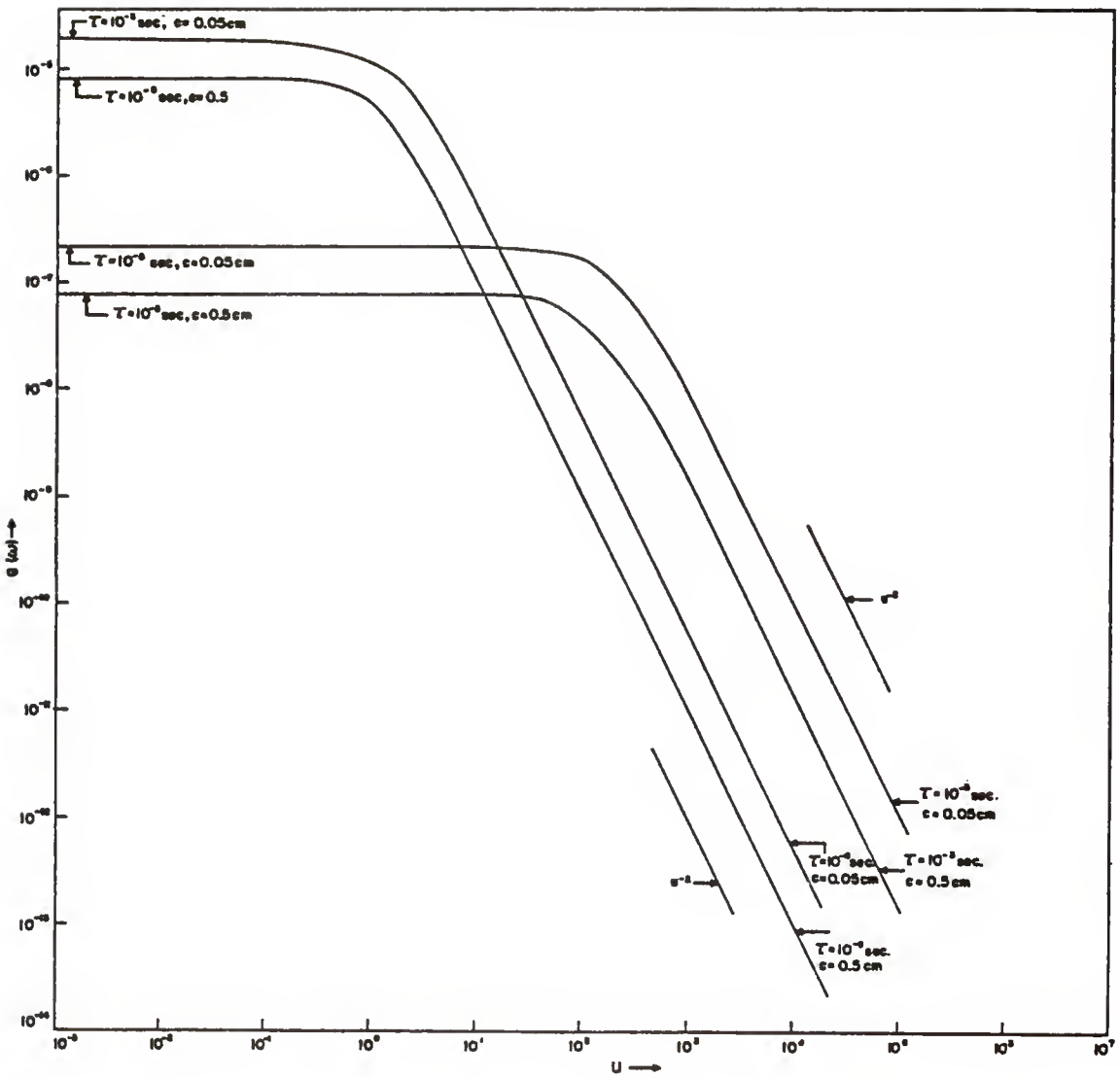


Fig. 9

Noise Spectrum for Nonembedded Bar

(B=0.5 cm for all plots,

$\tau=10^{-6}$sec.for upper two plots and $\tau=10^{-3}$sec.for lower two plots)

PART C
NOISE DUE TO SURFACE NOISE SOURCES

CHAPTER 8
NOISE DUE TO STOCHASTIC BOUNDARY CONDITIONS

8.1 Introduction

In this chapter we investigate the noise problem with the surface Langevin noise source. In this case we have the stochastic boundary condition

$$ma(\mathbf{r}_0,t) = \zeta(\mathbf{r}_0,t), \qquad r_0 \in \text{surface} \tag{8.1.1}$$

where m is a linear surface operator. We have mentioned earlier in chapter 2 that—because of the superposition principle-- we can separately consider the noise sources associated with the bulk noise source ξ and the surface noise source ζ. We also assume that these noise sources are uncorrelated, otherwise there will be a contribution S_a^{cross}, due to the correlation of these noise sources.

8.2 Summary of Previous Work

During past years, 1/f behavior of the noise spectrum in thin metallic films has been attributed to temperature fluctuations. Recently van Vliet et al.[14] considered temperature-fluctuation noise of thin films supported by a substrate. They considered noise due to (i) surface heat transfer and (ii) thermal diffusion in the sample and substrate. It was shown that the surface source results in a large 1/f range, the level of which decreases with increasing heat conductivity of the substrate. At high frequencies the noise goes

over into $1/f^2$ behavior. The thermal diffusion noise is shown to result in a $1/f^{1/2}$ spectrum at lower frequencies, going over to a $1/f^{3/2}$ spectrum at high frequencies. This noise also decreases with increasing heat conductivity of the substrate. However, for most practical cases the 1/f range is drowned by the $1/f^{1/2}$ thermal diffusion noise. In this chapter we investigate a similar problem, associated with g-r noise. There is again a volume and a surface contribution. The aim of this section is to determine (a) whether the surface contribution gives a 1/f range; (b) whether this noise can be dominant over the bulk g-r noise.

8.3 Noise Caused by Surface Generation-Recombination

In this section we investigate the noise spectrum resulting from the noise source ζ. We will consider three models.

First, we consider a bulk semiconductor with surface recombination at both boundaries. The fluctuations in the whole sample are of interest. Next we consider a MOSFET, with an accumulation layer serving as the channel, with surface recombination both on the oxide side and at the bulk semiconductor side. The noise concerns here the carrier population in the carrier enriched channel only; we take the mobility and lifetime to be uniform over the entire channel-semiconductor region. Finally, we consider a MOSFET model with surface recombination at the oxide side and reflection of the current at the semiconductor side, with the mobility and carrier lifetime in the channel being different from those in the bulk semiconductor.

8.3.1 Surface g-r Noise in a Bulk Semiconductor

The transport problem is described by the following equation

$$\frac{\partial \Delta p(x,t)}{\partial t} - D\frac{\partial^2 \Delta p(x,t)}{\partial x^2} + \frac{\Delta p(x,t)}{\tau} = \xi(x,t) \tag{8.3.1}$$

with boundary conditions

$$D \left.\frac{\partial \Delta p}{\partial x}\right|_{x=0} - S\Delta p\Big|_{x=0} = \zeta(t), \tag{8.3.2a}$$

$$D \left.\frac{\partial \Delta p}{\partial x}\right|_{x=W} + S\Delta p\Big|_{x=W} = \zeta(t). \tag{8.3.2b}$$

For simplicity we assume the surface recombination velocity S to be equal at both surfaces. (S is identical to σ, used in Chapter 6).

The noise due to the bulk source ξ was solved for already in the preceding chapters; so we set it $\equiv 0$ here, since we want to find the noise due to the source ζ, whose spectrum was given in chapter 3:

$$S_\zeta = 4M\alpha S p_0/A \tag{8.3.3}$$

where p_0 is the average concentration of holes, $\alpha = \langle \Delta p^2 \rangle / p_0$, M is the modulation factor ≥ 1, and A is the surface area.

To solve for the noise, we make a Fourier transform of Δp and of ζ

$$\Delta p(x,t) = \sum_{n=-\infty}^{\infty} \hat{p}(\omega_n, x)\exp(i\omega_n t), \qquad \omega_n = \frac{2\pi n}{T}, n=0,\pm 1,\ldots \tag{8.3.4}$$

$$\zeta(t) = \sum_{n=-\infty}^{\infty} \hat{\zeta}(\omega_n)\exp(i\omega_n t). \tag{8.3.5}$$

The connection between $\hat{p}$ and $\hat{\zeta}$ being linear, there exists a surface Green's function H, such that[24]

$$\hat{p}(\omega_n,x) = H(x,0,i\omega_n)\hat{\zeta}(\omega_n). \tag{8.3.6}$$

For the connection between the spectra of Δp and ζ we let $T \to \infty$ and $\omega_n \to \omega = 2\pi f$; then, since $S_{\Delta p}(x,x',f) = \lim_{T\to\infty} 2T\langle\hat{p}(\omega_n,x)\hat{p}^*(\omega_n,x')\rangle_s$ and $S_\zeta = \lim_{T\to\infty} 2T\langle\hat{\zeta}(\omega_n)\hat{\zeta}^*(\omega_n)\rangle_s$, we have

$$S_{\Delta p}(x,x',f) = H(x,0,i\omega)\, H(x',0,-i\omega) S_\zeta, \tag{8.3.7}$$

for any $x,x' \geq 0$. The spatially averaged carrier-fluctuation noise for the sample of thickness W is

$$S_{\Delta\bar{p}}(f) = \frac{1}{W^2}\int_0^W\int_0^W H(x,0,i\omega)\, H(x',0,-i\omega)\, dx\, dx'\, S_\zeta \tag{8.3.8}$$

or

$$S_{\Delta\bar{p}}(f) = \frac{S_\zeta}{S^2}\left|\frac{S}{W}\int_0^W H(x,0,i\omega)\,dx\right|^2. \tag{8.3.9}$$

Thus for the noise spectrum we must know the response function

$$f(i\omega) = \frac{S}{W}\int_0^W H(x,0,i\omega)\, dx. \tag{8.3.10}$$

To find the surface Green's function H, we define the Green's function of the Laplace transformed transport equation for $s = i\omega$ (for equation (8.3.1)) as

$$i\omega G(x,x',i\omega) - \frac{\alpha\partial^2 G(x,x',i\omega)}{\partial x^2} = \delta(x-x'), \tag{8.3.11}$$

where

$$\alpha = \frac{Di\omega}{i\omega + \frac{1}{\tau}}$$

subject to homogeneous boundary corresponding to equations (8.3.2a) and (8.3.2b)

$$D \frac{\partial G(x,x',i\omega)}{\partial x}\Big|_{x=0} - SG(x,x',i\omega)\Big|_{x=0} = 0 \tag{8.3.12a}$$

and

$$D \frac{\partial G(x,x',i\omega)}{\partial x}\Big|_{x=W} + sG(x,x',i\omega)\Big|_{x=W} = 0. \tag{8.3.12b}$$

The one dimensional Green's function defined above satisfies the condition of continuity at $x = x'$ and the jump condition

$$\frac{\partial G(x,x',i\omega)}{\partial x}\Big|_{x=x'-\varepsilon}^{x=x'+\varepsilon} = -\frac{1}{\alpha}. \tag{8.3.13}$$

We also define the adjoint Green's function

$$-i\omega\tilde{G}(x,x',i\omega) - \alpha\frac{\partial^2\tilde{G}(x,x',i\omega)}{\partial x^2} = \delta(x-x'). \tag{8.3.14}$$

The regular and adjoint Green's functions are related by the reciprocity relation

$$\tilde{G}^*(x,x',i\omega) = G(x',x,i\omega). \tag{8.3.15}$$

The Fourier-transform of equation (8.3.1) is

$$i\omega\hat{p}(\omega,x) - \alpha\frac{d^2p(\omega,x)}{dx^2} = 0. \tag{8.3.16}$$

According to Morse and Feshbach[26], we multiply equation (8.3.14) by $\hat{p}$ and equation (8.3.16) by $\tilde{G}^*$, then we subtract these equations and integrate the result over all x from 0 to W. We thus obtain, with the scalar product being defined as

$$(f,g) = \int_0^W dx\, fg^*, \tag{8.3.17}$$

$$\left(\frac{d^2\hat{p}(\omega,x)}{dx^2}, \tilde{G}(x,x',i\omega)\right) - \left(\hat{p}(\omega,x), \frac{d^2\tilde{G}(x,x',i\omega)}{dx^2}\right) = \frac{1}{\alpha}\hat{p}(\omega,x'), \quad (8.3.18)$$

for $0 \le x' \le W$.

The Green's theorem for particle diffusion, considering the Dirichlet-Neumann boundary conditions at $x=0$ and at $x=W$, is

$$\left(\frac{d^2f}{dx^2},h\right) - \left(f, \frac{d^2h}{dx^2}\right) = h^*\left[\frac{df}{dx} + \frac{S}{D}f\right]_{x=W} - f\left[\frac{dh^*}{dx} + \frac{S}{D}h^*\right]_{x=W}$$

$$-h^*\left[\frac{df}{dx} - \frac{S}{D}f\right]_{x=0} + f\left[\frac{dh^*}{dx} - \frac{S}{D}h^*\right]_{x=0} . \quad (8.3.19)$$

Substitution of equation (8.3.19) into equation (8.3.18), noticing the boundary conditions, yields

$$\frac{\tilde{G}^*(0,x',i\omega)}{D} = \frac{1}{\alpha}\hat{p}(\omega,x'); \quad 0 \le x' \le W . \quad (8.3.20)$$

This gives for the Fourier coefficient $\hat{p}$

$$\hat{p}(\omega,x') = \frac{\alpha}{D}\tilde{G}^*(0,x',i\omega)\hat{\zeta} . \quad (8.3.21)$$

Further, we replace x' by x in equation (8.3.21) and use the reciprocity relation given in equation (8.3.15); we then obtain

$$\hat{p}(\omega,x) = \frac{\alpha}{D}G(x,0,i\omega)\hat{\zeta} . \quad (8.3.22)$$

Comparing equations (8.3.6) and (8.3.22), we identify the surface Green's function in terms of the bulk Green's function

$$H(x,0,i\omega) = \frac{\alpha}{D}G(x,0,i\omega). \quad (8.3.23)$$

Now that we have established the relation between the surface and the bulk Green's function, we proceed to find the bulk Green's function. Consider the homogeneous equation corresponding to (8.3.11)

$$i\omega G(x,x',i\omega) - \alpha \frac{\partial^2 G(x,x',i\omega)}{\partial x^2} = 0, \tag{8.3.24}$$

valid for $x \neq x'$. At $x = x'$ we require continuity of G and the jump condition (8.3.13). We denote the Green's function for different regions as follows

$$G_a \quad \text{for} \quad 0 < x < x' \tag{8.3.25}$$

$$G_b \quad \text{for} \quad x' < x < W \tag{8.3.26}$$

So we have

$$G_a = A\sinh\frac{x}{L} + B\cosh\frac{x}{L} \tag{8.3.27}$$

$$G_b = C\sinh\frac{W-x}{L} + E\cosh\frac{W-x}{L} \tag{8.3.28}$$

where

$$L = \sqrt{\frac{\alpha}{i\omega}} = \sqrt{\frac{D}{i\omega + 1/\tau}} \ . \tag{8.3.29}$$

For G_a, the b.c. at $x=0$ gives

$$D\left(\frac{A}{L}\cosh\frac{x}{L} + \frac{B}{L}\sinh\frac{x}{L}\right)\Big|_{x=0} - S\left(A\sinh\frac{x}{L} + B\cosh\frac{x}{L}\right)\Big|_{x=0} = 0$$

or $\frac{DA}{L} - SB = 0$ gives

$$B = \frac{AD}{SL}, \tag{8.3.30}$$

and for G_b, the b.c. at $x = W$ gives

$$-D\left(\frac{C}{L}\cosh\frac{W-x}{L} + \frac{E}{L}\sinh\frac{W-x}{L}\right)\Big|_{x=W} + S\left(C\sinh\frac{W-x}{L} + E\cosh\frac{W-x}{L}\right)\Big|_{x=W} = 0$$

$$\text{or} \quad \frac{-DC}{L} + SE = 0 \quad \text{or} \quad E = \frac{CD}{SL} \quad . \tag{8.3.31}$$

Substituting B and E gives for G_a and G_b

$$G_a = A\left(\sinh\frac{x}{L} + \frac{D}{SL}\cosh\frac{x}{L}\right) \tag{8.3.32}$$

$$G_b = C\left(\sinh\frac{W-x}{L} + \frac{D}{SL}\cosh\frac{W-x}{L}\right) \quad . \tag{8.3.33}$$

Now, continuity at $x = x'$ gives

$$A\left(\sinh\frac{x'}{L} + \frac{D}{SL}\cosh\frac{x'}{L}\right) - C\left(\sinh\frac{W-x'}{L} + \frac{D}{SL}\cosh\frac{W-x'}{L}\right) = 0 \tag{8.3.34}$$

and the jump condition (8.3.13) gives

$$A\left(\cosh\frac{x'}{L} + \frac{D}{SL}\sinh\frac{x'}{L}\right) + C\left(\cosh\frac{W-x'}{L} + \frac{D}{SL}\sinh\frac{W-x'}{L}\right) = \frac{L}{\alpha}. \tag{8.3.35}$$

The system determinant is

$$D_r = \begin{bmatrix} \left(\sinh\frac{x'}{L} + \frac{D}{SL}\cosh\frac{x'}{L}\right) & -\left(\sinh\frac{W-x'}{L} + \frac{D}{SL}\cosh\frac{W-x'}{L}\right) \\ \left(\cosh\frac{x'}{L} + \frac{D}{SL}\sinh\frac{x'}{L}\right) & \left(\cosh\frac{W-x'}{L} + \frac{D}{SL}\sinh\frac{W-x'}{L}\right) \end{bmatrix}$$

$$= \left(1 + \frac{D^2}{S^2L^2}\right)\sinh\frac{W}{L} + \frac{2D}{SL}\cosh\frac{W}{L} \quad . \tag{8.3.36}$$

For A and C we find

$$A = \frac{L}{\alpha}\left(\sinh\frac{W-x'}{L} + \frac{D}{SL}\cosh\frac{W-x'}{L}\right)/D_r \tag{8.3.37}$$

$$C = \frac{L}{\alpha}\left(\sinh\frac{x'}{L} + \frac{D}{SL}\cosh\frac{x'}{L}\right)/D_r \quad . \tag{8.3.38}$$

So, finally,

$$G_b(x,x';i\omega) = \frac{L}{\alpha D_r}\left(\sinh\frac{W-x}{L} + \frac{D}{SL}\cosh\frac{W-x}{L}\right)\left(\sinh\frac{x'}{L} + \frac{D}{SL}\cosh\frac{x'}{L}\right) \quad (8.3.39)$$

$$H = \frac{\alpha}{D}G_b(x,0,i\omega) = \frac{1}{S\,D_r}\left(\sinh\frac{W-x}{L} + \frac{D}{SL}\cosh\frac{W-x}{L}\right). \quad (8.3.40)$$

The response function is

$$f(i\omega) = \frac{1}{WD_r}\int_0^W \left(\sinh\frac{W-x}{L} + \frac{D}{SL}\cosh\frac{W-x}{L}\right) dx \quad (8.3.41)$$

$$= \frac{-L}{WD_r}\left(1 - \cosh\frac{W}{L} - \frac{D}{SL}\sinh\frac{W}{L}\right). \quad (8.3.42)$$

The spectrum is, according to equations (8.3.3) and (8.3.10)

$$\frac{4M\alpha p_0}{sA}\,|f(i\omega)|^2 . \quad (8.3.43)$$

Denoting by $\overline{S} = |f(i\omega)|^2$, we find for the normalized spectrum

$$\overline{S}(\omega) = \frac{1}{W^2}\left|\frac{L\left(1 - \cosh\frac{W}{L} - \frac{D}{SL}\sinh\frac{W}{L}\right)}{\left(1+\frac{D^2}{S^2L^2}\right)\sinh\frac{W}{L} + \frac{2D}{SL}\cosh\frac{W}{L}}\right|^2 . \quad (8.3.44)$$

We investigate the following ranges.

(a) At sufficiently low frequencies, $|W/L| \ll 1$, so that

$\sinh\frac{W}{L} \simeq \frac{W}{L}$, $\quad \cosh\frac{W}{L} \simeq 1$.

Then

$$\overline{S}(\omega) \simeq \left|\frac{\sqrt{D(i\omega+\frac{1}{\tau})}}{S+2\sqrt{D(i\omega+\frac{1}{\tau})}}\right|^2 \xrightarrow{\omega\to 0} \left(\frac{1}{\sqrt{\frac{\tau_d\tau}{\tau_s^2}}+2}\right)^2 , \quad (8.3.45)$$

where, as in chapter 6, we defined $\tau_s = W/S$, $\tau_d = W^2/D$, so that $S^2/D = \tau_d/\tau_s{}^2$.

(b) At high frequencies $|W/L| \gg 1$, so that

$$\sinh \frac{W}{L} \simeq \cosh \frac{W}{L} \simeq e^{W/L} \quad ;$$

we then obtain

$$\overline{S}(\omega) \simeq \frac{1}{W^2} \left| \frac{(S+\sqrt{D(i\omega+\frac{1}{\tau})}\,)\,\sqrt{D/(i\omega+\frac{1}{\tau})}}{S+2\sqrt{D(i\omega+\frac{1}{\tau})} + (\frac{D}{S})\sqrt{i\omega+\frac{1}{\tau}}} \right|^2$$

$$\simeq \frac{1}{(\omega\tau_s)^2} \quad . \tag{8.3.46}$$

The spectrum closely resembles a Lorentzian.

8.3.2 Surface g-r Noise in a MOSFET: Uniform Parameters

The band diagram for a MOSFET with inversion layer is given in fig. 9. The quantum mechanical carrier density distribution in a MOSFET was computed by Hsing et al.[30], and is reproduced in fig.10. This distribution differs strongly from the classical distribution using the profile in fig.9 and Fermi-Dirac statistics. For our purpose we shall approximate the distribution of fig.10 by a rectangle such that

$$n = n_s, \quad 0 \le x \le W_1 \quad \text{(Surface region)}$$

$$n = n_b, \quad W_1 \le x \le W_2. \quad \text{(Bulk region)}$$

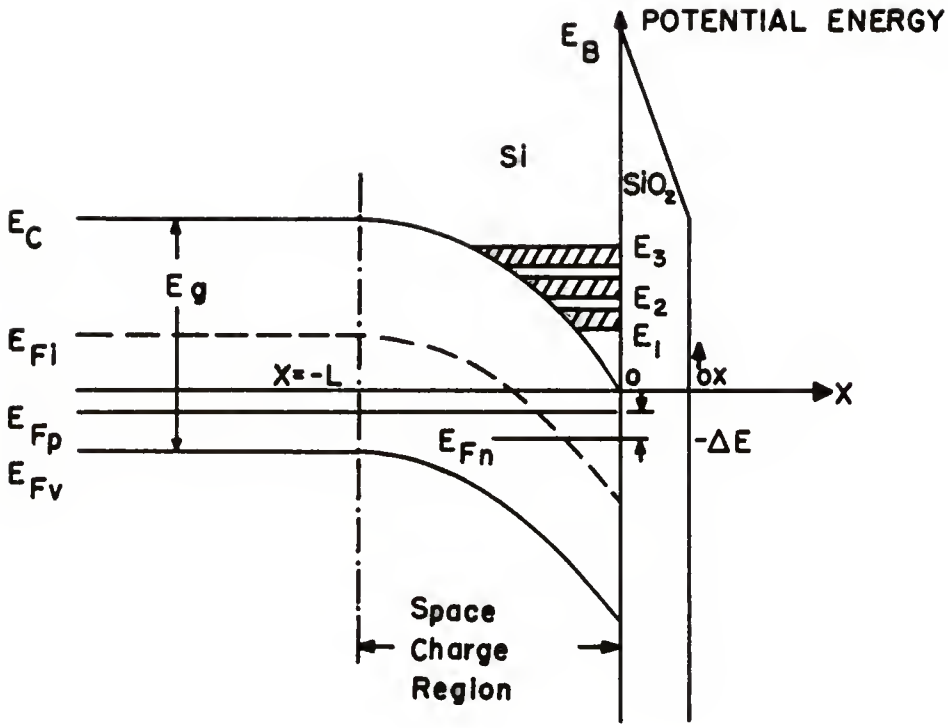


Fig. 9

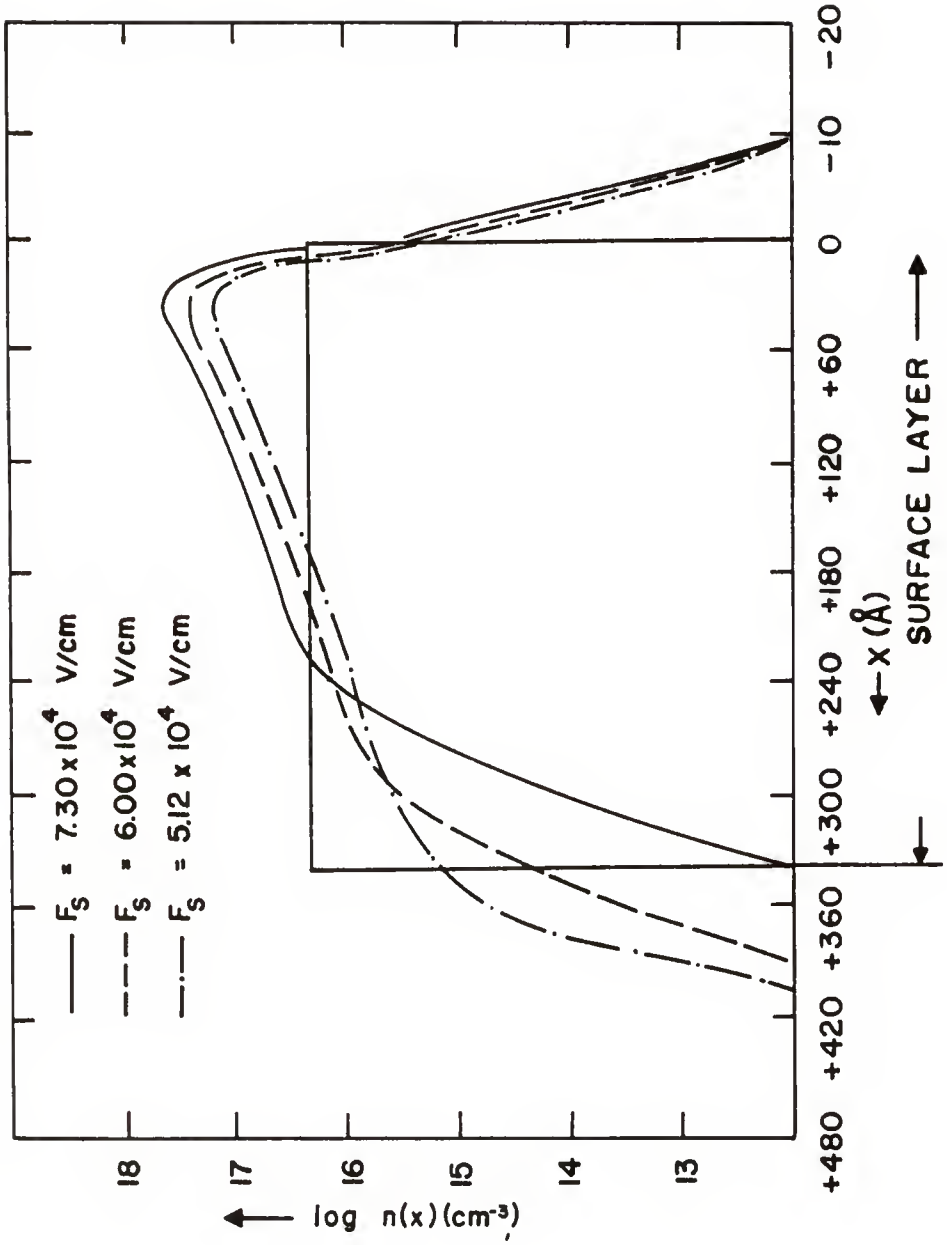
Energy band diagram for a MOSFET

(After Hsing et al. Ref.30)

Fig. 10

The quantum mechanical carrier distribution in a MOSFET

(After Hsing et al., ref.30)


F_s = 7.30 x 10^4 V/cm
F_s = 6.00 x 10^4 V/cm
F_s = 5.12 x 10^4 V/cm
log n(x) (cm^{-3})
18
17
16
15
14
13
+480
+420
+360
+300
+240
+180
+120
+60
0
-10
-20
X (Å)
SURFACE LAYER

Though the carrier density in the bulk n_b is much less than the channel density n_s, carriers diffuse into the bulk and undergo bulk recombination and surface recombination at the other end. In the present subsection, we assume for simplicity that the surface recombination velocity in the oxide and at the free surface of the semiconductor are the same. Also, we assume μ, D, and τ to be the same in the channel as in the bulk. The transport equation and b.c. are then the same as in the previous subsection, substituting $p \to n$, and with S_ζ

$$S_\zeta = 4M\alpha n_s S/A. \tag{8.3.47}$$

The surface noise may exceed the bulk noise in this model since $n_s \gg n_b$ and probably $M \gg 1$.

For the surface Green's function we have the same as before; in the response function we must now integrate from 0 to $W_1 \ll W$, $W_1+W_2 = W$. Thus

$$f(i\omega) = \frac{1}{D_r W_1} \int_0^{W_1} \left(\sinh \frac{W-x}{L} + \frac{D}{SL}\cosh \frac{W-x}{L}\right) dx \tag{8.3.48}$$

$$= \frac{-L}{W_1 D_r} \left(\cosh \frac{W_2}{L} + \frac{D}{SL}\sinh \frac{W_2}{L} - \cosh \frac{W}{L} - \frac{D}{SL}\sinh \frac{W}{L}\right). \tag{8.3.49}$$

Using equation (8.3.43), we find for the normalized spectrum

$$\bar{S}(\omega) = \frac{1}{W_1^2} \left| \frac{L\left(\cosh \frac{W_2}{L} + \frac{D}{SL}\sinh \frac{W_2}{L} - \cosh \frac{W}{L} - \frac{D}{SL}\sinh \frac{W}{L}\right)}{\left(1+ \frac{D^2}{S^2L^2}\right)\sinh \frac{W}{L} + \frac{2D}{SL}\cosh \frac{W}{L}} \right|^2 \tag{8.3.50}$$

This can also be rewritten as

$$\bar{S}(\omega) = \frac{1}{W_1^{\,2}} \left| \frac{2L[\sinh(\frac{W_2}{L} + \frac{W_1}{2L})\sinh\frac{W_1}{2L} + \frac{D}{SL}\cosh(\frac{W_2}{L} + \frac{W_1}{2L})\sinh\frac{W_1}{2L}]}{(1+\frac{D^2}{S^2L^2})(\sinh\frac{W_1}{L}\cosh\frac{W_2}{L} + \cosh\frac{W_1}{L}\sinh\frac{W_2}{L}) + \frac{2D}{SL}(R)} \right|^2 \tag{8.3.51}$$

where $\quad R = (\cosh\frac{W_1}{L}\cosh\frac{W_2}{L} + \sinh\frac{W_1}{L}\sinh\frac{W_2}{L})$.

In this result we assume $W_2 \gg W_1$. We discuss the following cases.

(a) At sufficiently low frequencies $\left|\frac{W_1}{2L}\right| \ll 1$, $\left|\frac{W_2}{L}\right| \ll 1$, so that

$$\sinh\frac{W_1}{2L} \approx \frac{W_1}{2L} \quad \text{and} \quad \cosh\frac{W_2}{L} \approx 1.$$

Thus for the normalized spectrum, noting $W_2 \gg W_1$,

$$\bar{S}(\omega) \underset{\omega\to\text{very low}}{\simeq} \frac{1}{W_1^{\,2}} \left| \frac{2L(\frac{W_2}{L}\frac{W_1}{2L} + \frac{D}{SL}\frac{W_1}{2L})}{(1+\frac{D^2}{S^2L^2})(\frac{W_1}{L} + \frac{W_2}{L} + \frac{2D}{SL})} \right|^2 \tag{8.3.52}$$

$$\simeq \left| \frac{(W_2 + \frac{D}{S})}{(1+\frac{D^2}{S^2L^2})(W_2 + \frac{2D}{S})} \right|^2 \tag{8.3.53}$$

$$= \frac{\text{constant}}{(1+\frac{D}{S^2}\frac{1}{\tau})^2 + \omega^2\frac{D^2}{S^4}} \quad . \tag{8.3.54}$$

We have $D/S^2 = \tau_s^{\,2}/\tau_d$; this gives

$$\overline{S}(\omega) \underset{\omega\to 0}{\simeq} \frac{\text{constant}}{(1+\frac{\tau_s^{\,2}}{\tau_d\tau})^2} . \tag{8.3.55}$$

(b) At intermediate frequencies $\left|\frac{W_1}{2L}\right| \ll 1$ and $\left|\frac{W_2}{L}\right| \gg 1$,

thus

$$\sinh\frac{W_1}{2L} \approx \frac{W_1}{2L}, \quad \sinh\frac{W_2}{L} \approx \cosh\frac{W_2}{L} \approx e^{W_2/L} .$$

The normalized spectrum is

$$\overline{S}(\omega) = \frac{1}{W_1^{\,2}} \left| \frac{2L(e^{W_2/L}\frac{W_1}{2L} + \frac{D}{SL}e^{W_2/L}\frac{W_1}{2L})}{(1+\frac{D^2}{S^2L^2})e^{W_2/L} + \frac{2D}{SL}e^{W_2/L}} \right|^2 \tag{8.3.56}$$

$$= \left| \frac{(1+\frac{D}{SL})}{(1+\frac{D^2}{S^2L^2}+\frac{2D}{SL})} \right|^2 .$$

For $\omega \gg 1/\tau$, the term D^2/S^2L^2 dominates the denominator and D/SL dominates the numerator. One then finds

$$\overline{S}(\omega) \underset{\omega\to\text{int.}}{=} \frac{\tau_d}{\tau_s^{\,2}\omega} . \tag{8.3.57}$$

The noise spectrum shows 1/f behavior at intermediate frequencies. In MOSFETs it has been shown experimentally that 1/f behavior is observed to very high frequencies.

(c) At much higher frequencies $\left|\frac{W_1}{2L}\right| \gg 1$, $\left|\frac{W_2}{L}\right| \gg 1$, and $\sinh \frac{W_1}{2L} \approx \cosh \frac{W_1}{2L} \approx e^{W_1/2L}$, $\sinh \frac{W_2}{L} \approx \cosh \frac{W_2}{L} \approx e^{W_2/L}$. Thus we obtain for the spectrum

$$\bar{S}(\omega) \approx \frac{1}{W_1^2} \left| \frac{L(1+\frac{D}{SL})}{(1+\frac{D^2}{S^2L^2})} \right|^2 \tag{8.3.58}$$

$$\bar{S}(\omega) \underset{\omega \to \text{very high}}{\approx} \frac{S^2}{D^2}|L^4| = \frac{S^2}{W_1^2\omega^2} . \tag{8.3.59}$$

The noise spectrum goes as $\frac{1}{\omega^2}$ at very high frequencies. The $1/\omega$ and $1/\omega^2$ ranges meet at the break frequency ω_0, such that $\omega_0 = \frac{\tau_s^2 S^2}{\tau_d W_1^2} = \frac{D}{W_1^2}$, providing $\omega_0 \gg \frac{1}{\tau}$, where we used $\tau_s = \frac{W}{S}$, $\tau_d = \frac{W^2}{D}$.

Let $D = 25\ cm^2/sec$ and $W_1 = 400\ \text{Å}$, see Fig.10. This yields $\omega_0 = 1.6\times10^{12}$ rad/sec or $f_0 \approx 25\times10^{10}$ hz. Clearly we have always that $\omega_0 \gg \frac{1}{\tau}$. The above break frequency is to all likelihood never observable. If $\tau = 10^{-4}$ sec., then the spectrum goes as $1/f$ for

$$2\times10^3 < f < 25\times10^{10}.$$

Hence, the $1/f$ range can indeed be many decades!

8.3.3 Surface g-r Noise in a MOSFET, Nonuniform Parameters

We now consider a MOSFET model where the material parameters μ,D and τ (carrier lifetime) in the channel are different from those in the bulk semiconductor. We assume that in the bulk semiconductor

these parameters are μ', D' and τ'. We have

for $0 < x < W_1$, the parameters are μ, D, τ

for $W_1 < x < W$, the parameters are μ', D', τ'.

The transport equation is

$$\frac{\partial}{\partial t}\Delta n(x,t) - D\frac{\partial^2 \Delta n(x,t)}{\partial x^2} + \frac{\Delta n(x,t)}{\tau} = \xi(x,t). \tag{8.3.60}$$

For the boundary conditions we assume in this subsection:

$$D\left.\frac{\partial \Delta n}{\partial x}\right|_{x=0} - S\Delta n\Big|_{x=0} = \zeta(t) \tag{8.3.61a}$$

$$D'\left.\frac{\partial \Delta n}{\partial x}\right|_{x=W} = 0. \tag{8.3.61b}$$

So $S=0$ at the free end of the bulk, which means that the carriers are reflected at $x=W$.

This model is mathematically identical to the model considered by van Vliet et al.[14] for the surface heat transfer noise, provided we make the following substitutions in their equations (2.17)-(2,19):

$$\sigma \to D, \quad g \to S, \quad \sigma' \to D'.$$

We also note that in our case α is different from that in their paper, viz.,

$$\alpha = \frac{Di\omega}{i\omega + 1/\tau}, \qquad \alpha' = \frac{D'i\omega}{i\omega + 1/\tau'} \tag{8.3.62}$$

and

$$L = \sqrt{\frac{\alpha}{i\omega}}, \qquad L' = \sqrt{\frac{\alpha'}{i\omega}}. \tag{8.3.63}$$

We find from their paper for the response function (after the necessary substitutions)

$$f(i\omega) = \frac{\frac{L}{W_1}\left[\tanh\frac{W_1}{L} + \frac{D'L}{DL'}\tanh\frac{W_2}{L'}\,\frac{\cosh(\frac{W_1}{L})-1}{\cosh(\frac{W_1}{L})}\right]}{\left[(\tanh\frac{W_1}{L} + \frac{D}{SL})\frac{D'L}{DL'}\tanh\frac{W_2}{L'} + 1 + \frac{D}{SL}\tanh\frac{W_1}{L}\right]} \tag{8.3.64}$$

We now discuss the different cases.

(a) At very low frequencies $\left|\frac{W_1}{L}\right| \ll 1$, $\left|\frac{W_2}{L'}\right| \ll 1$ and we have

$$\tanh\frac{W_1}{L} \approx \frac{W_1}{L}, \quad \tanh\frac{W_2}{L'} \approx \frac{W_2}{L'}, \quad \cosh\frac{W_1}{L} \approx 1.$$

For the normalized spectrum we obtain

$$\overline{S}(\omega) \approx \left|\frac{1}{1+\frac{W_1W_2D'}{DLL'} + \frac{D'W_2}{SLL'} + \frac{D}{SL}\frac{W_1}{L}}\right|^2 \tag{8.3.65}$$

$$\overline{S}(\omega) \underset{\omega\to\text{very low}}{\approx} 1 . \tag{8.3.66}$$

(b) At intermediate frequencies $\left|\frac{W_1}{L}\right| \ll 1$, $\left|\frac{W_2}{L'}\right| \gg 1$ so that

$$\tanh\frac{W_1}{L} \approx \frac{W_1}{L}, \quad \tanh\frac{W_2}{L'} \approx 1, \quad \cosh\frac{W_1}{L} \approx 1 .$$

We get for the spectrum

$$\overline{S}(\omega) \approx \left|\frac{1}{\frac{W_1D'}{DL'} + \frac{D'}{SL'} + 1 + \frac{D}{SL}\frac{W_1}{L}}\right|^2 . \tag{8.3.67}$$

In the denominator D'/SL' dominates. Thus, for $\omega \gg 1/\tau$,

$$\overline{S}(\omega) = \frac{\tau_d'}{\tau_s'^2\omega} , \tag{8.3.68}$$

where $\tau_d' = W^2/D' \approx W_2^2/D'$, $\tau_s' = W/S \approx W_2/S$.

Equation (8.3.68) is of the same form as (8.3.5), except that we need the bulk parameters.

The spectrum shows 1/f behavior at intermediate frequencies

(c) At high frequencies $\left|\frac{W_1}{L}\right| \gg 1$, $\left|\frac{W_2}{L'}\right| \gg 1$, and we have $\tanh \frac{W_1}{L} \approx 1$, $\tanh \frac{W_2}{L'} \approx 1$, $\cosh \frac{W_1}{L} \approx e^{W_1/L}$.

For the spectrum we find

$$\overline{S}(\omega) \approx \left| \frac{\frac{L}{W_1}(1+ \frac{LD'}{DL'})}{(1+ \frac{D}{SL})\frac{D'L}{DL'} + 1 + \frac{D}{SL}} \right|^2 . \tag{8.3.69}$$

Further we have that $|L| \ll D/S, |L'| \ll D'/S$The spectrum becomes

$$\overline{S}(\omega) \underset{\omega\to\text{very high}}{\approx} \frac{S^2}{W_1^2\omega^2} . \tag{8.3.70}$$

This is the same as in subsection 8.2.2, see equation (8.3.59). At high frequencies the spectrum goes as $\frac{1}{\omega^2}$. Again, the high frequency range is near unobtainable, since the break frequency is of the order of 10^{10} hz.

We see that even though the material parameters are not uniform in the model of this subsection the asymptotic or limiting behavior of the noise spectrum is the same as in the model with uniform parameters.

Also, the noise ranges are apparently insensitive to the boundary condition (or to the value of the surface recombination velocity) at $x = W$. The magnitude of the noise in the 1/f range in the present model is higher than in the uniform parameter model, since $D(\text{volume})/D(\text{surface}) = \mu(\text{volume})/\mu(\text{surface}) \approx 5$ (see Hsing et al.[31]). The diffusion through the channel, represented by the term $(D/SL)(W_1/L)$ in equation (8.3.67) plays no role since the channel is very thin.

Altogether, the models presented in subsections 8.3.2 and 8.3.3 give a very realistic 1/f range, extending over many decades. This noise is due to the surface source, which stems from the recombination-through "fast" surface states. This is in contrast to McWhorter's[12] model, in which the 1/f noise is attributed to fluctuations through tunneling into spatially distributed "slow" surface states. The latter can explain 1/f noise for very low frequencies, whereas in our model the 1/f range begins at $\omega \approx 1/\tau$, τ being the channel carrier lifetime. This is a drawback compared to McWhorter's model[12]. However, since in MOSFETs the 1/f noise often is observable for very high frequencies, at least part of the 1/f noise could well be attributed to the mechanisms considered in this chapter.

APPENDIX

This Appendix lists the following computer programs used to calculate the noise spectra.

A.1 Nonsymmetrical embedded bar for diffusion noise source, (section 5.3.1).

For "P" source and different geometries the following changes should be made:

A.1.a. For "P" source, remove the term 'TEMP' from numerator of F(L), just above statement '10 CONTINUE', (Section 5.2.1).

A.1.b. For square disc use W1BAR = 0.1, W2BAR = 0.1, W3BAR = 100, (Section 5.3.2).

A.1.c. For square bar use W1BAR = .01, W2BAR = 10.0, W3BAR = 10.0, (Section 5.3.3).

A.1.d. For cube use W1BAR = 1.0, W2BAR = 1.0, W3BAR = 1.0, (Section 5.3.4).

A.2 Nonsymmetrical nonembedded bar, (Section 7.3).

A.3 Symmetrical embedded sphere for $\eta = 0$, (Section 4.4.1).

A.4 Symmetrical embedded sphere for $\eta \neq 0$, (Section 4.4.2).

A.5 Symmetrical nonembedded sphere for $\eta = 0$, (Section 6.3.1).

A.6 Symmetrical nonembedded sphere for $\eta \neq 0$, (Section 6.3.3).

A.1 Nonsymmetrical Embedded Bar (Diffusion Noise Source)

```
C
C--     TYPE DECLARATION
C
        DIMENSION F(101),W(101)
C
C--     DEFINE CONSTANTS
C
        W1BAR=1. 0E-2
        W2BAR=1. 0
        W3BAR=1. 0E2
        WMIN=1. 0E-3
        WMAX=1. 0E3
        NPTS=13
        DW=(WMAX/WMIN)**(1. 0/(NPTS-1))
C
C--     IF ICNTRL = 1, RANGE OF INTEGRATION (RI) IS 0TO RANGE1 IN
C--                    NDIV STEPS,
C--               = 2, RI IS 0 TO RANGE1 IN NDIV STEPS + RANGE1 TO
C--                    10*RANGE1 in NDIV STEPS,
C--               = 3, RI IS 0 TO RANGE1 IN NDIV STEPS + RANGE1 TO
C--                    10*RANGE1 IN NDIV STEPS + 10*RANGE1 TO
C--                    100*RANGE1 IN NDIV STEPS.
C
        ICNTRL=2
        NDIV=20
        RANGE1=1. 0
        STEP=RANGE1/NDIV
C
C--     COMPUTE THE FREQUENCY RESPONSE
C
        DO 40 L=1,NPTS
        W(L)=WMIN*DW**(L-1)
        F(L)=0. 0
        ZT=-STEP
        DZT=STEP
        DO 35  IZT=1, ICNTRL
        DZT=DZT*10, 0**(IZT-1)
        NDIV1=NDIV
        DO 30  K=1,NDIV1
        ZT=ZT+DZT
        ET=-STEP
        DET=STEP
        DO 25  IET=1, ICNTRL
        DET=DET*10. 0**(IET-1)
        DO 20  J=1,NDIV1
        ET=ET+DET
        ZI=-STEP
        DZI=STEP
        DO 15  IZI=1,ICNTRL
        DZI=DZI*10. 0**(IZI-1)
```

```
      DO 10  I=1,NDIV1
      ZI=ZI+DZI
      TEMP=W1BAR*ZT**2+W2BAR*ET**2+W3BAR*ZI**2
      F(L) =F(L)+SINCSQ(ZT/2)*SINSQ(ET/2)*SINCSQ(ZI/2)*TEMP/
     $         (TEMP**2+W(L)**2)
 10   CONTINUE



 15   ZI=ZI+DZI
 20   CONTINUE
 25   ET=ET+DET
 30   CONTINUE
 35   ZT=ZT+DZT
 40   CONTINUE
C
C--   NORMALIZE ABOUT THE CENTER FREQUENCY
C
      F1=F(NPTS/2+1)
      DO 50 L=1,NPTS
      F(L)=F(L)/F1
      F(L)=10.0*ALOG10(F(L))
 50   CONTINUE
C
C--   PRINT THE RESULTS
C
      WRITE(6,60) (I,W(I),F(I),I=1,NPTS)
 60   FORMAT(10X,I5,2E17,6/)
      STOP
      END


C
C
      FUNCTION  SINCSQ(X)
      IF  (X.EQ.0.0) SINCSQ=1.0
      IF  (X.EQ.0.0) RETURN
      TEMP=(SIN(X)/X)**2
      SINCSQ=TEMP
      RETURN
      END
```

A.2 Nonsymmetrical Nonembedded Bar

```
C
C--       TYPE DECLARATION
C
          DIMENSION  ETA(40),ZETA(40),FSQ(40,40),
         $                   OMG(101),G(101)
          REAL       K1,K2,NO
          COMPLEX    FIJ
C
C--       DEFINE ABSOLUTE CONSTANTS
C
          PI=3.141592654
          B=0.5
          C=0.5
          D=25.
          S=500
          TOU=1. OE=3
          N=20
          WMIN=1. E-2
          WMAX=1. E+4
          NPTS=13.
C
C--       DEFINE DEPENDENT CONSTANTS
C
          K1=S*B/D
          K2=S*C/D
          CO=1.0/TOU
          C1=D/B**2
          C2=D/C**2
          C3=TOU
          DW=(WMAX/WMIN)**(1.0/(NPTS-1))
C
C--       SOLVE TRANSCENDENTAL EQUATIONS FOR ZETA(K) & ETA(K), K=1,2,3,...
C
          CALL SOLTRN(ZETA,K1,N)
          CALL SOLTRN(ETA,K2,N)
C
C--       EVALUATE THE MATRIX FSQ(I,J)
C
          DO 10, I=1,N
          Z2=2.0*ZETA(I)
          DO 10 J=1,N
          E2=2.0*ETA(J)
          FIJ=4.0*SIN(ZETA(I))*SIN(ETA(J))/
         $    CSQRT(CMPLX(ZETA(I)*ETA(J)*(Z2+SIN(Z2))*(E2+SIN(E2)),0.0))
          FSQ(I,J)=CABS(FIJ)**2
 10       CONTINUE
```

```
C
C--       EVALUATE THE FREQUENCY RESPONSE
C
          DO 30 K=1,NPTS
          OMG(K)=WMIN*DW**(K-1)
          SUM=0.0
          DO 20 I=1,N
          DO 20 J=1,N
          TEMP=1+C1*C3*ZETA(I)**2+C2*C3*ETA(J)**2
          SUM=SUM+FSQ(I,J)*TEMP/(OMG(K)**2+TEMP**2)
 20       CONTINUE
          G(K)=SUM
 30       CONTINUE
C
C--       PRINT THE RESULTS
C
          WRITE(6,40)(OMG(I),G(I),I=1,NPTS)
 40       FORMAT (5X,2E20.5/)
          STOP
          END
C
C
C--       SOLTRN SOLVES THE TRANSCENDENTAL EQUATION X*TAN(X)=C FOR
C--         X(K), K=1,2,3,...,N.
C
          SUBROUTINE  SOLTRN(X,C,N)
          DIMENSION  X(N),CO(2),XO(2),WS(10)
          EXTERNAL TRAN
          PI=3.14
          EPS=1. OE-6
          XO(1)=.25
          CO(1)=C
          IT=50
          DO 10  I=1,N
          CALL  ZSYSTM(TRAN,EPS,6,1,XO,IT,WS,CO,IERR)
          X(I)=XO(1)
 10       XO(1)=XO(1)+PI
          RETURN
          END
```

A.3 Symmetrical Embedded Sphere (η=0)

```
C
C--       TYPE DECLARATION
C
          DIMENSION W(101),F(101),B(101),C(101),D(101),E(101),G(101),H(101),
         $A(100)
C
C--       DEFINE CONSTANTS
C
          WMIN=1.E-3
          WMAX=1.E5
          NPTS=17
          DW=(WMAX/WMIN)**(1.0/(NPTS-1))
C
C--       COMPUTE THE FREQUENCY RESPONSE
C
          DO 40  L=1,NPTS
          W(L)=WMIN*DW**(L-1)
          F(L)=0.0
          B(L)=SQRT(2*W(L))
          C(L)=SQRT(.5*W(L))
          A(L)=(2.*W(L)*(SIN(B(L))+COS(B(L))))
          D(L)=(2.*(COS(B(L))-SIN(B(L))))+(4.*B(L)*COS(B(L)))
          E(L)=(2.*(COS(C(L))+SIN(C(L))))-(2.*B(L)*COS(C(L)))
          G(L)=(2.*(COS(C(L))-SIN(C(L))))+(2.*B(L)*COS(C(L)))
          H(L)=(EXP(-B(L))*A(L)+D(L))
          H(L)=-2.+2.*W(L)+H(L)
          F(L)=F(L)+(H(L)/(B(L)**5))
C
 40       CONTINUE
C
C--       PRINT AND PLOT THE RESULTS
C
          WRITE (6,60) (I,W(I),F(I),I=1,NPTS)
 60       FORMAT(10X,I5,2E17.6/)
          STOP
          END
```

A.4 Symmetrical Embedded Sphere ($\eta \neq 0$)

```
C
C--       TYPE DECLARATION
C
          DIMENSION W(101),F(101),B(101),C(101),D(101),E(101),G(101),H(101),
         $A(100),P(100),Q(100),C1(100),G1(100),Q1(100)
C
C--       DEFINE CONSTANTS
C
          WMIN=1.E-1
          WMAX=1.E7
          ET=.01
          NPTS=17
          DW=(WMAX/WMIN)**(1.0/(NPTS-1))
C
C--       COMPUTE THE FREQUENCY RESPONSE
C
          DO 40 L=1,NPTS
          W(L)=WMIN*DW**(L-1)
          F(L)=0.0
          W(L)=W(L)*ET
          B(L)=((SQRT(2.))*(SQRT(ET+SQRT(ET**2+W(L)**2))))
          D(L)=((SQRT(2.))*(SQRT(-ET+SQRT(ET**2+W(L)**2))))
          A(L)=1./((B(L)**2+D(L)**2)**5)
          C(L)=((10.*B(L)**2*D(L)**3)-(5.*B(L)**4*D(L))-(D(L)**5))
          G(L)=((2.*B(L)*D(L))+ ((2*B(L)*D(L)*EXP(-B(L))*COS(D(L)))
         $(4.*D(L)*EXP(-B(L))*COS(D(L))))
         $+(EXP(-B(L))*SIN(D(L))*(D(L)**2-B(L)**2-4.*B(L)-4.)))
          C1(L)=(-(B(L)**5-10.*B(L)**3*D(L)**2+5.*B(L)*D(L)**4))
          G1(L)=((B(L)**2-D(L)**2)+((2.*D(L)*B(L)*EXP(-B(L))*SIN(D(L)))
         $+(4.*D(L)*EXP(-B(L))*SIN(D(L))))
         $+(EXP(-B(L))*COS(D(L))*(B(L)**2-D (L)**2+4.+4.*B(L)))-4.)
          H(L)=(((B(L)**2-D(L)**2))/(3.*((B(L)**2+D(L)**2)**2)))
          Q(L) =(C(L)*G(L))
          Q1(L)=(C1(L)*G1(L))
          P(L)=(Q(L)+Q1(L))
          E(L)=(A(L)*P(L))
          F(L)=F(L)+E(L)+H(L)
 40       CONTINUE
C
C--       PRINT THE RESULTS
C
          WRITE(6,60) (I,W(I),F(I),I=1,NPTS)
 60       FORMAT(10X,I5,2E17,6/)
          STOP
          END
```

A.5 Symmetrical Nonembedded Sphere (η=0)

```
C
C--       TYPE DECLARATION
C
          DIMENSION W(101),F(101),B(101),C(101),D(101),E(101),H(101),
         $A(101),A1(101),D1(101)
C
C--       DEFINE CONSTANTS
C
          WMIN=1.E-3
          WMAX=1.E3
          NPTS=13
          G=0.1
          DW=(WMAX/WMIN)**(1.0/(NPTS-1))
C
C--       COMPUTE THE FREQUENCY RESPONSE
C
          DO 40  L=1,NPTS
          W(L)=WMIN*DW**(L-1)
          F(L)=0.0
          E(L)=((-SIN(E(L))*COSH(E(L)))+(E(L)*COS(E(L))*COSH(E(L)))
         $+(E(L)*SIN(E(L))*SINH(E(L))))
          B(L)=((COS(E(L))*SINH(E(L)))+(E(L)*SIN(E(L))*SINH(E(L)))
         $-(E(L)*COS(E(L))*COSH(E(L))))
          C(L)=((COS(E(L))*SINH(E(L)))+(E(L)*SIN(E(L))*SINH(E(L)))
         $-(E(L)*COS(E(L))*COSH(E(L)))-((COS(E(L)*SINH(E(L)))/G))
          D(L)=((-SIN(E(L))*COSH(E(L)))+(E(L)*COS(E(L))*COSH(E(L)))
         $+(E(L)*SIN(E(L))*SINH(E(L)))+((SIN(E(L))*COSH(E(L)))/G))
          A1(L)=3.*((A(L)*D(L))+(B(L)*C(L)))
          D1(L)=((W(L)**2)*G*((C(L)**2)+(D(L)**2)))
          F(L)=F(L)+(A1(L)/D1(L))
 40       CONTINUE
C
C--       PRINT THE RESULTS
C
          WRITE(6,60) (I,W(I)),F(I),I=1,NPTS)
 60       FORMAT(10X,I5,2E17,6/)
          STOP
          END
```

A.6 Symmetrical Nonembedded Sphere ($\eta \neq 0$)

```
C
C--       TYPE DECLARATION
C
          DIMENSIONW(99),F(99),B(99),C(99),E(99),A1(99),H(99),B1(99),
         $F1(99),D1(99),A(99),P(99),Q(99),G1(99),P1(99),C1(99),G1(99)
C
C--       DEFINE CONSTANTS
C
          ET=0.1
          WMIN=1.E-3
          WMAX=1.E5
          NPTS=17
          G=0.1
          DW=(WMAX/WMIN)**(1.0/(NPTS-1))
C
C--       COMPUTE THE FREQUENCY RESPONSE
C
          DO 40 L=1,NPTS
          W(L)=WMIN*DW**(L-1)
          F(L)=0.0
          H(L)=((.707) *(SQRT(ET+SQRT(ET**2+W(L)**2))))
          E(L)=((.707) *(SQRT(-ET+SQRT(ET**2+W(L)**2))))
          A(L)=((-SIN(E(L))*COSH(H(L)))+(E(L)*COS(E(L))*COSH(H(L)))
         $+(H(L)*SIN(E(L))*SINH(H(L))))
          B(L)=((COS(E(L))*SINH(H(L)))+(E(L)*SIN(E(L))*SINH(H(L)))
         $-(H(L)*COS(E(L))*COSH(H(L))))
          D(L)=((COS(E(L))*SINH(H(L)))+(E(L)*SIN(E(L))*SINH(H(L)))
         $-(H(L)*COS(E(L))*COSH(H(L)))-((COS(E(L))*SINH(H(L)))/G))
          C(L)=((-SIN(E(L))*COSH(H(L)))+(E(L)*COS(E(L))*COSH(H(L)))
         $+H(L)*SIN(E(L))*SINH(H(L)))+((SIN(E(L))*COSH(H(L)))/G))
          P1(L)=((H(L)**2)-(E(L)**2))
          C1(L)=((P1(L)*C(L))-(2.*H(L)*E(L)*D(L)))
          G1(L)=((P1(L)*D(L))+(2.*H(L)*E(L)*C(L)))
          Q(L)=((((H(L)**2)*C1(L))+((E(L)**2)*C1(L)))**2)
          Q1(L)=((((H(L)**2)*G1(L))+((E(L)**2)*G1(L)))**2)
          D1(L)=G*(Q(L)+Q1(L))
          A1(L)=((P1(L)*C1(L))-(2.*H(L)*E(L)*G1(L)))
          B1(L)=((P1(L)*G1(L))+(2.*H(L)*E(L)*C1(L)))
          P(L)=G*((C1(L)+A1(L))+(G1(L)*B1(L)))
          F1(L)=-3.*((A(L)*A1(L))+(B(L)*B1(L)))
          F(L)=F(L)+((P(L)+F1(L))/(D1(L)))
 40       CONTINUE
C
C--       PRINT THE RESULTS
C
          WRITE(6,60) (I,W(I),F(I),I=1,NPTS)
 60       FORMAT(10X,I5,2E17,6/)
          STOP
          END
```

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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